

Correspondence Theorem between Holomorphic Discs and Tropical Discs on K3 Surfaces

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Abstract

In this paper, we prove that the open Gromov-Witten invariants defined in [15] on K3 surfaces satisfy the Kontsevich-Soibelman wall-crossing formula. One application is that the open Gromov-Witten invariants coincide with the weighted counting of tropical discs. This generalizes the corresponding theorem on toric varieties [21][22] to K3 surfaces.

1 Introduction

The celebrated Strominger-Yau-Zaslow conjecture [25] suggested that Calabi-Yau manifolds admits special Lagrangian fibration near large complex limit points. Furthermore, the special Lagrangians are expected to collapse to the base affine manifolds at the limit [13]. Holomorphic curves under this limit are conjectured to converge to some 1-skeletons, known as tropical curves, on the base affine manifold. Ideally, this can reduce some enumerative problems on Calabi-Yau manifolds to the countings of the tropical curves, which are closer to combinatoric problems. However, the great picture is not generally carried out due to the involvement of Calabi-Yau metric, yet has no explicit expression so far. Instead, Mikhalkin [21] started the realm of tropical geometry by establishing the counting of holomorphic curves and tropical curves on toric surfaces. Later, the correspondence theorem is generalized to all toric manifolds [22].

For understanding mirror symmetry from the SYZ point of view, the author defined an version of open Gromov-Witten invariants [15], naively count the number of holomorphic discs with boundaries on special Lagrangian torus fibres of SYZ fibration. The open Gromov-Witten invariants $\tilde{\Omega}(\gamma; u)$

of relative class γ locally is an invariant with respect to the special Lagrangian boundary condition u . There are some real codimension one walls on the base of the fibration known as the walls of marginal stability. An explicit example is constructed that when the special Lagrangian boundary conditions vary across a wall of marginal stability, they can bound new holomorphic discs which only appear on one side of the wall. As a consequence, the counting invariant $\tilde{\Omega}(\gamma; u)$ jumps when the special Lagrangian boundary condition varies across the wall. This is similar to the wall-crossing phenomenon of generalized Donaldson-Thomas invariants [12]. Under certain primitive conditions, the wall-crossing formula is calculated in [15].

On the other hand, the author developed the notion of tropical discs and the corresponding counting invariants on elliptic K3 surfaces [15]. The tropical discs counting invariants satisfy the Kontsevich-Soibelman wall-crossing formula. In particular, a relative class can be realized as a tropical discs when the associated open Gromov-Witten invariants are nonzero. The main goal of the paper to show that the open Gromov-Witten invariants is indeed the same as the weighted tropical discs counting.

The paper is arranged as follows: In section two, we review the homological algebra including the notion of Maurer-Cartan equations and their properties. In section three, we define the tropical discs counting as weighted count of tropical discs and study its properties. The generating functions of the tropical discs counting are related to the slab functions in Gross-Seibert program [9] or Kontsevich-Soibelman transformation in [11]. In particular, they satisfy the Kontsevich-Soibelman wall-crossing formula. Moreover, one can consider the q -deformed the Kontsevich-Soibelman algebra which becomes non-commutative. There is a corresponding way to q -deform the tropical disc counting invariants and leads to a refined tropical discs invariant.

In section four, we first review the Floer theory on HyperKähler manifolds with special Lagrangian fibration. The hyperKähler condition makes the special Lagrangian fibres which bound holomorphic discs project to affine lines on the base. Then we use the idea of family Floer homology, which is originally advocated by Fukaya [5], to study the wall-crossing phenomenon of the open Gromov-Witten invariants. The similar method is also used to construct the mirror functor for the Lagrangian torus fibration [26][1]. The result of our paper can be view as the enumerative counterpart. More precisely, we study the Maurer-Cartan elements associated to the A_∞ structure of the special Lagrangian fibres which can be identified as the cohomology of the torus fibres. When we vary the special Lagrangian boundary conditions in a 1-parameter family, the A_∞ structures are related via pseudo-isotopies.

Moreover, pseudo-isotopies naturally identifies the Maurer-Cartan elements. The identification records the information of holomorphic discs appear in the 1-parameter family and not necessarily coincide with the identification via parallel transport. We match the identification the pseudo-isotopies with the Kontsevich-Soibelman transformation associated to certain tropical discs, including the wall-crossing formula and the computation the contribution from the initial discs. In particular, this leads to the proof of the main theorem of the paper.

Theorem 1.1. (*=Theorem 6.28*) *Let γ be a relative class and u does not fall on the wall of marginal stability of γ , then the open Gromov-Witten invariants and the weighted tropical discs counting are well-defined and coincide, i.e.*

$$\tilde{\Omega}(\gamma; u) = \tilde{\Omega}^{trop}(\gamma; u).$$

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2 Preliminary of Algebraic Framework

The section is a review of homological algebra in [6][3].

2.1 A_∞ -Structures and A_∞ Homomorphisms

Let $\Lambda^\mathbb{C}$ be the complex Novikov ring, namely

$$\Lambda^\mathbb{C} = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{C}, \lambda_i \in \mathbb{C}, \text{ and } |\lambda_i| \text{ increasing, } \lim |\lambda_i| = +\infty \right\}.$$

We will denote its maximal ideal by Λ_+ . We will always assume \mathcal{S} is a sector with angle less than π . We define

$$\Lambda^\mathcal{S} := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \in \Lambda \mid \text{Arg} \lambda_i \in \mathcal{S}, \lim |\lambda_i| = \infty \right\}.$$

There is a natural valuation $val : \Lambda^{\mathbb{C}}(\text{or } \Lambda^{\mathcal{S}}) \rightarrow \mathbb{R}_{\geq 0}$ given by

$$val(\sum_i a_i T^{\lambda_i}) = |\lambda_0|.$$

We will use $F^{\lambda} \Lambda^{\mathbb{C}}(\text{or } F^{\lambda} \Lambda^{\mathcal{S}})$ to denote the subset of $\Lambda^{\mathbb{C}}(\text{or } \Lambda^{\mathcal{S}})$ consists of elements with val less than λ for a $\lambda > 0$. When the sector \mathcal{S} degenerates to a ray, then $\Lambda^{\mathcal{S}}$ is naturally identified with the standard Novikov ring Λ . Notice that val gives a filtration on $\Lambda^{\mathbb{C}}$ does not with respect to the multiplication. However, we have the following substitute.

Lemma 2.1. *Let Γ be a lattice and \mathcal{S} be a sector with angle less than π . Assume that $Z : \Gamma \rightarrow \mathbb{C}$ be a homomorphism such that $\Gamma_{\mathcal{S}} := Z^{-1}(\mathcal{S})$ has finite intersection with $B_0(\lambda)$, for every $\lambda > 0$. Then*

1. *for every $\lambda > 0$, we have $B_0(\lambda) \cap \hat{\Gamma}_{\mathcal{S}}$ is finite, where $\hat{\Gamma}_{\mathcal{S}}$ is the monoid generated by $\Gamma_{\mathcal{S}}$.*
2. *given $\lambda' > 0$, then there exists $\lambda > 0$ such that and a subset A of $Z(\Gamma_{\mathcal{S}})$ which does not intersect with $B_0(\lambda)$. Then there exists $\lambda' > 0$, independent of A , such that the monoid generated by A has no intersection with $B_0(\lambda')$.*

Proof. It suffices to prove the case when the sector \mathcal{S} has angle $\pi/2$, since the topology will be equivalent. We will put \mathcal{S} as the first quadrant. For any $\lambda > 0$, $\hat{x} \in B_0(\lambda) \cap \hat{\Gamma}_{\mathcal{S}}$ then $\hat{x} = \sum x_i$ with $x_i \in B_0(\lambda) \cap \Gamma_{\mathcal{S}}$. Since the set $B_0(\lambda) \cap \Gamma_{\mathcal{S}}$ is finite, set λ' be the smallest x or y coordinate of elements in $B_0(\lambda) \cap \Gamma_{\mathcal{S}}$. Then there are at most $2[\frac{\lambda}{\lambda'}] + 2$ of such \hat{x} . This proves the first part of the lemma. The second lemma is straight forward after we transform \mathcal{S} to a sector with angle π . \square

Definition 2.2. *Let G be a monoid with an evaluation map $\omega : T \rightarrow \mathbb{R}_{\geq 0}^1$ such that*

$$|\omega^{-1}[0, \lambda)| < \infty,$$

for every $\lambda \in \mathbb{R}_{\geq 0}$.

Let C be a graded vector space over Λ and let

$$BC[1] = \bigoplus_k B_k C[1], \quad B_k C[1] = C[1]^{\otimes k}$$

be its associated bar complex.

¹In the later application, we will take ω be either the symplectic form the central charge defined in (4).

Definition 2.3. We say C admits an filtered gapped A_∞ algebra structure if there exists a monoid G and a sequence of homomorphisms $\{m_{k,\gamma}\}_{k \geq 0, \gamma \in G}$ of degree $+1$

$$m_{k,\gamma} : B_k C[1] \rightarrow C[1]$$

such that $\hat{d} \circ \hat{d} = 0$, where \hat{d} is the coderivation induced from

$$m_k = \sum_{\gamma \in G} m_{k,\gamma} T^{\omega(\gamma)}, k \geq 0.$$

Definition 2.4. Let $(C, m_{k,\gamma})$ and $(C', m'_{k,\gamma})$ be two gapped filtered A_∞ algebras. The sequence of sequence of \mathbb{R} -linear $f_{k,\gamma} : B_k C[1] \rightarrow C'[1]$ be maps of degree 0, for $k \in \mathbb{N}$ and $\gamma \in G$ is an A_∞ homomorphism if $\hat{d}' \circ \hat{f} = \hat{f} \circ \hat{d}$.

2.2 Canonical Model and Homotopy Lemma

Let (C, m_k) be an filtered G -gapped A_∞ -algebra and $m_{1,0}^2 = 0$. Let H be its $m_{1,0}$ -cohomology and the projection $p : C \rightarrow H$ is of degree zero. Assume that there exists a deformation retract from C to H , namely, a pair (H, ι) where

1. $\iota : H \rightarrow C$ of degree zero is an injective homomorphism of vector spaces such that

$$d\iota = 0, p d = 0 \text{ and } p \iota = \text{Id}.$$

2. The linear map $G : C \rightarrow C$ of degree -1 such that

$$\text{Id} - \iota p = -(dG + Gd). \tag{1}$$

Theorem 2.5. [6] Under the assumption above, there exists an A_∞ structure on H and an invertible A_∞ homomorphism $f : H \rightarrow C$.

Proof. The proof is standard and we include a sketch for introducing certain notation use later. We construct the canonical A_∞ structure and A_∞ homomorphism as follows:

Definition 2.6. Let $Gr(k, \gamma)$ be the set of decorated rooted tree T with the following properties:

1. The vertices $C_0(T)$ are the disjoint union of exterior vertices $C_0^{ext}(T)$ and interior vertices $C_0^{int}(T)$. The root $v_0 \in C_0^{ext}(T)$ and $|C_0^{ext}(T)| = k + 1$.

2. Each exterior vertex has valency 1.
3. Each interior vertex $v \in C_0^{int}(T)$ is labeled by $\gamma_v \in G$. The valency of v is greater than 2 if $\gamma_v = 0$.
4. The sum of labels of interior vertices is γ .

With the above definition, an edge is called an exterior edge if it is adjacent to an exterior vertex. Otherwise, it is called an interior edge.

We associate a homomorphism for each tree $T \in Gr(k, \gamma)$ as follows: For the edge adjacent to the root, we insert the projection operator p . For other edges adjacent to an exterior vertex, we insert the operator ι . For each interior edge, we insert the operator G . We insert the operator m_{k_v, γ_v} for each interior vertex, where $k_v = \text{val}(v) - 1$. Let $m_{k, \gamma}^{can}$ denotes summation of the composition operator in above way, summing over all possible decorated rooted trees in $Gr(k, \gamma)$. Then it is straight forward to check that $\{m_{k, \gamma}\}$ defines a filtered G -gapped A_∞ structure on H .

Moreover, we replace operator associate to the edge adjacent to the root by G and similar composition operator $f_{k, \gamma}$ defines an A_∞ -homomorphism from H to C . \square

2.3 Pseudo-Isotopies of A_∞ Algebras

Definition 2.7. Two gapped filtered A_∞ algebras $(C, m_{k, \gamma}^0)$ and $(C, m_{k, \gamma}^1)$ are pseudo-isotopy if there exists a gapped filtered A_∞ structure on $(C^\infty([0, 1]_t, C), \hat{m}_{k, \gamma})$ restricting to $(C, m_{k, \gamma}^0)$ (and $(C, m_{k, \gamma}^1)$) as $t = 0$ (and $t = 1$ respectively).

Remark 2.8. For $\mathbf{x}_i = x_i(t) + dt \wedge y_i(t) \in C^\infty([0, 1], C)$, we write

$$\hat{m}_k(\mathbf{x}_1, \dots, \mathbf{x}_k) = x(t) + dt \wedge y(t),$$

where

$$\begin{aligned} x(t) &= m_k^t(x_1(t), \dots, x_k(t)), \\ y(t) &= c_k^t(x_1(t), \dots, x_k(t)) \\ &\quad - \sum_{i=1}^k (-1)^* m_{k, \gamma}^t(x_1(t), \dots, x_{i-1}(t), y_i(t), x_{i+1}(t), \dots, x_k(t)), \end{aligned}$$

if $(k, \gamma) \neq (1, 0)$ and

$$y(t) = \frac{d}{dt} x_1(t) + m_{1, 0}^t(y_1(t)).$$

Then the A_∞ relation of \hat{m}_k is equivalent to the following relations:

1. $m_{k,\gamma}^t$ and $\mathfrak{c}_{k,\gamma}^t$ are smooth in t^2 .
2. For any fixed t , $(C, m_{k,\gamma}^t)$ defines a gapped filtered A_∞ structure.
- 3.

$$\begin{aligned} & \frac{d}{dt} m_{k,\gamma}^t(x_1, \dots, x_k) \\ & + \sum (-1)^* \mathfrak{c}_{k_1,\gamma_1}^t(x_1, \dots, m_{k_2,\gamma_2}^t(x_i, \dots), \dots, x_k) \\ & - \sum m_{k_1,\gamma}^t(x_1, \dots, \mathfrak{c}_{k_2,\gamma_2}^t(x_i, \dots), \dots, x_k) = 0, \end{aligned}$$

where the sum is all possible $i, (k_1, \gamma_1)$ and (k_2, γ_2) such that $k_1 + k_2 = k + 1$ and $\gamma_1 + \gamma_2 = \gamma$.

4. $\frac{d}{dt} m_{k,0}^t = 0$ and $\mathfrak{c}_{k,0}^t = 0$ for every k .

Theorem 2.9. [3] *If there exists a pseudo-isotopy between two (filtered) A_∞ algebra $(C, m_{k,\gamma}^0)$ and $(C, m_{k,\gamma}^1)$, then there exists an invertible A_∞ homomorphism between them.*

Proof. We will need the construction of the A_∞ homomorphism for later usage, so here we sketch the construction the A_∞ homomorphism.

Let $Gr'(k, \gamma)$ be the set of trees T in $Gr(k, \gamma)^3$ together with a time allocation τ (See [6][3]). We will define $\mathfrak{c}(T)$ inductively on the number of exterior vertex of T .

1. If $|C_0^{int}(T)| = 0$, we set $\mathfrak{c}(T, \tau) = Id$.
2. If $|C_0^{int}(T)| = 1$, let $v \in C_0^{ext}(T)$ and $\text{val}(v) = k + 1$. Then we set

$$\mathfrak{c}(T, \tau)(x_1, \dots, x_k) = -\mathfrak{c}_{k,\gamma_v}^{\tau(v)}(x_1, \dots, x_k).$$

3. If $|C_0^{int}(T)| > 1$, let v be the vertex adjacent to the root. Assume that $(T_1, \tau_1), \dots, (T_k, \tau_k)$ are the subtrees (with time allocations) derived from deleting v and the root. Then we set

$$\mathfrak{c}(T, \tau) = -\mathfrak{c}_{k,\gamma_v}^{\tau(v)}(\mathfrak{c}(T_1, \tau_1) \otimes \dots \otimes \mathfrak{c}(T_k, \tau_k)).$$

²For the purpose of this paper, we will have $C = \Omega^*(L)$. and $m_{k,\gamma}^t, c_{k,\gamma}^t$ are smooth differential forms on $L \times [0, 1]$.

³Here we assume moreover that all the vertices with valency 1 are exterior vertices.

With the notation above, the A_∞ homomorphism from $(C, m_{k,\gamma}^{t_0})$ to $(C, m_{k,\gamma}^{t_1})$ is given by

$$\mathfrak{c}(k, \gamma) = \sum_{Gr(k, \gamma)} \mathfrak{c}(T),$$

where $\mathfrak{c}(T) = \int \mathfrak{c}(T, \tau) d\tau$ is the integral over the space of possible time allocation of T with values in $[t_0, t_1]$. We refer the detail of the proof to [3]. \square

Proposition 2.10. [3] *If two (cyclic) gapped filtered gapped A_∞ algebra are pseudo-isotopic to each other then so are their canonical models.*

2.4 Maurer-Cartan Equation and Maurer-Cartan Elements

Given a filtered gapped A_∞ algebra C , the evaluation map $\omega : C \rightarrow \mathbb{R}_{\geq 0}$ naturally extend to C , which we still denote it by ω . The kernel $\bar{C} = \omega^{-1}(0) \subseteq C$ is a vector space and we have the following definition.

Definition 2.11. 1. *Let C be an gapped filtered A_∞ algebra. The Maurer-Cartan equation of C is given by*

$$m(e^b) := m_0(1) + m_1(b) + \cdots + m_k(b, \cdots, b) + \cdots = 0.$$

We say $b \in C$ is a Maurer-Cartan element if $m(e^b) = 0$. We denote $\tilde{\mathcal{MC}}(C)$ to be the set of Maurer-Cartan elements.

2. *Let $b_0, b_1 \in \tilde{\mathcal{MC}}(C)$, we say b_0 is gauge equivalent to b_1 if there exists $b(t)$ and $c(t)$ such that*

$$(a) \ b(0) = b_0 \text{ and } b(1) = b_1.$$

(b)

$$\frac{d}{dt}b(t) + \sum_k m_k(b(t), \cdots, b(t), c(t), b(t), \cdots, b(t)) = 0$$

3. *The moduli space of Maurer-Cartan elements is defined by*

$$\mathcal{MC}(C) := \tilde{\mathcal{MC}}(C) / \sim,$$

where \sim is the gauge equivalence.

Proposition 2.12. [3] *If there exists a pseudo-isotopy $m_{k,\gamma}^t, \mathfrak{c}_{k,\gamma}^t$ between two filtered A_∞ algebras $(C, m_{k,\gamma}^0)$ and $(C, m_{k,\gamma}^1)$, then induces an isomorphism between Maurer-Cartan spaces*

$$F : \mathcal{MC}(C) \longrightarrow \mathcal{MC}(C')$$

$$b \longmapsto \sum_{k,\gamma} b + \mathfrak{c}(k, \gamma)(b, \dots, b)T^{\omega(\gamma)}.$$

Theorem 2.13. [3] *Let F_1 and F_2 be two pseudo-isotopies between filtered A_∞ algebras C and C' . Assume that there exists a pseudo-isotopy of pseudo-isotopy between F_1 and F_2 then*

$$(F_1)_* = (F_2)_* : \mathcal{MC}(C) \rightarrow \mathcal{MC}(C').$$

Example 2.14. *Let $L \subseteq X$ be a Lagrangian torus bounds no holomorphic discs. Then the A_∞ structure on $\Omega^*(L, \Lambda)$ reduces to a differential graded algebra,*

$$m_k = \begin{cases} \pm d & \text{if } k = 1 \\ \pm \int_L \cdot \wedge \cdot & \text{if } k = 2 \\ 0 & \text{otherwise,} \end{cases}$$

The associate Maurer-Cartan space is the de Rham equivalent classes of degree 1. The canonical model of $(\Omega^(L, \Lambda), m_k)$ is $(H^*(L, \Lambda), m_k^{\text{can}})$ which reduces to the cohomology ring. Namely,*

$$m_k^{\text{can}} = \begin{cases} \pm \langle, \rangle & \text{if } k = 2 \\ 0 & \text{otherwise.} \end{cases}$$

The Maurer-Cartan space of $H^(L, \Lambda)$ is the space of harmonic 1-forms on L with values in Λ_+ . The induced isomorphism from Theorem 2.5 between Maurer-Cartan spaces $\mathcal{MC}(\Omega^*(L, \Lambda))$ and $\mathcal{MC}(H^*(L, \Lambda))$ is the identification between harmonic forms and de Rham classes.*

3 Recap of HyperKähler Geometry

We will have a brief recap of the hyperKähler rotation which we will use through the whole paper. Let X be a complex surface with the holomorphic symplectic 2-form Ω . Assume that there exists a Kähler form ω in the Kähler class $[\omega]$ such that

$$\omega^2 = \frac{1}{2} \Omega \wedge \bar{\Omega},$$

then we say X is a hyperKähler surface.

Example 3.1. *Compact holomorphic symplectic manifolds are always hyperKähler by the theorem of Yau [27]. There are also non-compact examples such as Ooguri-Vafa space with Ooguri-Vafa metric [20].*

The pair (ω, Ω) can determine an S^1 -family⁴ of hyperKähler structures on the underline space of X with the Kähler form and holomorphic symplectic 2-form given by

$$\begin{aligned}\omega_\vartheta &:= -\text{Im}(e^{-i\vartheta}\Omega), \\ \Omega_\vartheta &:= \omega - i\text{Re}(e^{-i\vartheta}\Omega).\end{aligned}$$

for $\vartheta \in S^1$. We will denote the corresponding hyperKähler manifold by X_ϑ .

Let L_{u_0} be a holomorphic curve in X , thus $\Omega|_{L_{u_0}} = 0$. Then L_{u_0} is a special Lagrangian in X_ϑ in the sense that $\omega_\vartheta|_{L_{u_0}} = 0$ and $\text{Im}\Omega_\vartheta|_{L_{u_0}} = 0$. This procedure is called hyperKähler rotation. Let B be the linear system of L_{u_0} and B_0 parametrizes the smooth ones. Let u_0 be the point in B_0 corresponding to L_{u_0} and similarly for each $u \in B_0$, we denote the corresponding smooth holomorphic Lagrangians by L_u . After hyperKähler rotation L_u will become a special Lagrangian in X_ϑ . Before we can talk about tropical geometry, we have to introduce the integral affine structure on B_0 , for a given ϑ . Let $e_i \in H_2(X, L_{u_0}; \mathbb{Z}), i = 1, \dots, 2g$ be an integral basis. For any $u \in B_0$ near u_0 and a path $\phi(t)$ connecting u_0 and u , the functions

$$f_i(u) = \int_{\bar{e}_{i_u}} \text{Im}\Omega_\vartheta, i = 1, 2$$

give an integral affine structure on B_0 . We will use B_ϑ to denote the affine manifold (to distinguish the affine structures constructed from various ϑ). Here \bar{e}_{i_u} is the 2-chain which is the union of parallel transport of $\partial\gamma$ along the path ϕ . In the case $n = 2$, the affine structure is known as the complex affine coordinate in the context of mirror symmetry. The the affine functions f_i are real analytic (multi-value) functions on B_0 .

4 Tropical Geometry

Naively, tropical curves/discs are union of affine line segments on an integral affine manifold with the "balancing condition". In this section, we will

⁴It actually determines an S^2 -family of hyperKähler structures but we will only use the particular $S^1 \subseteq S^2$.

explore the tropical geometry on K3 surface (with special Lagrangian fibration and only I_1 -type singular fibres). It worth mentioning that the tropical curves/discs in this paper live on the base of special Lagrangian fibration. On the other hand, Gross-Seibert [9] considered the tropical curves on the dual intersection complex from a toric degeneration (see also [28] for the setting in non-Archimedean geometry). However, it is not clear if the base of the special Lagrangian fibration can be identified with the affine manifold in the Gross-Seibert program. This is due to the complexity of the details of Ricci-flat metric on K3 surfaces. It also worth mentioning that there is a tropical discs counting/tropical Donaldson-Thomas invariants defined in [15][14] by assigning initial conditions and use the expected wall-crossing formula to define the invariants. Here in this paper, we define the tropical discs counting as the weighted count of admissible tropical discs, including the ones passing through the singularities of the affine manifold.

Definition 4.1. *Let B be an affine 2-manifold with singularities Δ and with an integral structure on TB . In other words, there exists an integral affine structure on $B \setminus \Delta$. Assume that around each singularity of the affine structure the monodromy is conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ ⁵. Let B_0 be the complement of the singularities Δ . For a rooted connected graph T , we denote the set of vertices and edges by $C_0(T)$ and $C_1(T)$ respectively. A tropical curve (with stop u) on B is a 3-tuple (ϕ, T, w) where T is a rooted connected graph (with a root x), a weight function $w : C_1(T) \rightarrow \mathbb{N}$ and $\phi : T \rightarrow B$ is a continuous map such that*

1. $\phi(x) = u \in B_0$.
2. We allow G to have unbounded edges only when B is non-compact.
3. For any vertex $v \in C_0(T)$, the unique edge e_v closest to the stop is called the outgoing edge of v and write $w_v := w(e_v)$.
4. For each $e \in C_1(T)$, $\phi|_e$ is either an embedding of affine segment on B_0 or $\phi|_e$ is a constant map. Assume that $\phi(e) \not\subseteq \Delta$. Let $v_e \in \Gamma(e, \phi^*T_{\mathbb{Z}}B_0)$ be a primitive flat section. In the former case, we will further assume that v_e is in the tangent direction of $\phi(e)$ and pointing toward $\phi(x)$.
5. For each $v \in C_0(T)$, $v \neq x$ and $\text{val}(v) = 1$, we have $\phi(v) \in \Delta$. Moreover,

⁵We will discuss the tropical geometry with other singularities in the next paper [19].

- (a) If $\phi|_{e_v}$ is an embedding, then $\phi(e_v)$ in the monodromy invariant direction⁶.
- (b) Let $T' \subseteq T$ be a connected subtree contains v such that $\phi(T') = \phi(v)$, then
 - i. For e_v , we associate an primitive integral vector $v_{e_v} \in T_{\phi(v)+}B$ in the monodromy invariant direction such that $\phi(v)^+ = \text{Exp}_{\phi(v)}(\epsilon v_{e_v})$ for some $\epsilon > 0$.
 - ii. For each edge e in T' or edge adjacent to T' , we associate a primitive integral vector $e_v \in T_{\phi(v)+}B$.
 - iii. For each $v \in C_0(T') \cap C_0^{\text{ext}}(T)$, the balancing condition (2) holds at $T_{\phi(v)}^+B$.
 - iv. Let e be an edge adjacent to T' . Then v_e (up to parallel transport along the shortest path from $\phi(v)^+$ to $\phi(e)$) is the primitive tangent vector of $\phi(e)$.
- 6. For each $v \in C_0(T)$, $v \neq x$ and $\text{val}(v) = 2$, we have $\phi(v) \in \Delta$. Moreover,
 - (a) the edges e_v^+, e_v^- adjacent to v are not contracted by ϕ .
 - (b) $\phi(e_v^\pm)$ is in the monodromy invariant direction and the primitive integral vectors associated to e_v^\pm is monodromy invariant.
 - (c) $w(e_v^+) = w(e_v^-)$.
- 7. For each $v \in C_0(T)$, $\text{val}(v) \geq 3$, $\phi(v) \notin \Delta$, we have the following assumption: (balancing condition) Let e_1, \dots, e_n, e_v are the edges adjacent to v . Then

$$w_{e_v} v_{e_v} = \sum_i w_{e_i} v_{e_i} \in T_{\phi(v)}B. \quad (2)$$

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The main difference from the usual definition of tropical discs in the literature is that we allow an edge maps to a point and the affine structures can have singularities. The balancing condition (2) will make the following definition well-defined.

⁶Straight forward computation shows that the monodromy invariant direction of the affine structure is rational.

⁷By induction, all v_e are of rational slope.

Definition 4.2. Let $\phi : T \rightarrow B$ be a parametrized tropical rational curves (with stop x). Assume that $\text{val}(v) = 3$ for a vertex $v \neq x$. The multiplicity at such vertex v is given by

$$\text{Mult}_v(\phi) = w_1 w_2 |v_{e_1} \wedge v_{e_2}|, \quad (3)$$

where e_1, e_2 are two of the edges adjacent to v and $w_i = w(e_i)$. The last term $v_{e_1} \wedge v_{e_2}$ in (3) falls in $\wedge^2 T_{\phi(v)} B \cong \mathbb{Z}^8$ due to the integral structure of TB_0 .

Definition 4.3. A tropical disc is a tropical curve (w, T, ϕ) with stop $u \in B_0$ and T is a tree.

Naively, tropical discs with stop at u corresponds to holomorphic discs with boundary on L_u . One very different nature of tropical discs on K3 surfaces is the following observation, which is a direct consequence of Lemma 4.12.

Proposition 4.4. Every tropical discs in B is rigid.

This reflects the fact that the moduli space of holomorphic discs with boundary on special Lagrangian torus fibres have negative virtual dimension.

Since our goal is to define a tropical discs counting and compare it with open Gromov-Witten invariants, we first have to associate a relative class for each tropical disc. Tropical discs can only lift to relative cycles in X up to signs, which is related to the reality condition of open Gromov-Witten invariants. Thus to define a relative class for a tropical discs and for the defining the tropical discs counting invariants $\tilde{\Omega}^{\text{trop}}(\gamma; u)$ in Definition 4.8, we will equip X a Kähler class.

Definition 4.5. Let X be a K3 surface with holomorphic volume form Ω and $X \rightarrow B$ be an elliptic fibration with 24 singular fibres and fix a choice of Kähler class $[\omega]$. Fix $e^{i\vartheta} \in S^1$ and let B_ϑ the affine manifold with the affine structure given by the complex affine structure with singularities of X_ϑ .

1. Given a tropical disc $\phi : T \rightarrow B$ with stop at u on the integral affine manifold B_ϑ , we will associate it with a relative class⁹ as follows by induction on the number of singularities of affine structure ϕ hits: If

⁸We will replace $\phi(v)$ by $\phi(v)^+$ if $\phi(v) \in \Delta$.

⁹The class $[\phi]$ can also be understood as follows: one associate a cylinder (wrapping w_e times) by normal construction for each edge e of tropical disc. At each trivalent vertex, we glue in a pair of pant as local model. However, it is not clear to the author the suitable local model for vertex with higher valency.

the ϕ only hits only one singularity and has its stop at u , then let $[\phi] \in H_2(X, L_u)$ be the relative class of Lefschetz thimble such that $\text{Arg} \int_{[\phi]} \Omega > 0$. Otherwise let p be the internal vertex of ϕ closest to the root x and let ϕ_1, \dots, ϕ_s be the components of $\text{Im}(\phi) \setminus p$ containing an ingoing edge of p . By induction we already define $[\phi_i] \in H_2(X, L_p)$. Then $[\phi]$ is defined to be the parallel transport of $\sum_{i=1}^s [\phi_i]$ from $\phi(p)$ to u .

2. Given a tropical disc ϕ on B_ϑ , the central charge of ϕ is defined to be

$$Z_\phi = \int_{[\phi]} \Omega.$$

Definition 4.6. 1. Let $v_1, \dots, v_n \in M \cong \mathbb{Z}^2$ be primitive vectors (not necessarily distinct) and $\mathfrak{d}_{ij} \in M \otimes \mathbb{R}$ be the lines in the direction v_i with weight w_{ij} , $j = 1, \dots, l_i$. Assume that $w_{ij} \leq w_{ij'}$ if $j \leq j'$. We order \mathfrak{d}_{ij} such that $\mathfrak{d}_{i_1 j_1} < \mathfrak{d}_{i_2 j_2}$ if

- (a) $i_1 < i_2$ or
- (b) $i_1 = i_2$ and $j_1 < j_2$.

2. We say that the lines $\{\mathfrak{d}_{ij}\}$ are in the standard position if the intersection of $\mathfrak{d}_{i_1 j_1}$ and $\mathfrak{d}_{i_2 j_2}$ is on the far right side of the line \mathfrak{d}_{ij} if $\mathfrak{d}_{ij} > \mathfrak{d}_{i_1 j_1}$ and $\mathfrak{d}_{ij} > \mathfrak{d}_{i_2 j_2}$.
3. Let (ϕ, T, w) be a tropical curve in \mathbb{R}^2 with no contracted edges. We say (ϕ, T, w) is in the standard position with respect to $\{(v_i, w_{ij})\}$ if T has $|\sum_i l_i| + 1$ unbounded edges such that all except one unbounded edges are mapped into some \mathfrak{d}_{ij} with weight w_{ij} by ϕ . The exceptional unbounded edge has direction v and weight w such that $wv = \sum_{i,j} v_i w_{ij}$.

The following definition explains which tropical discs will contribute to the tropical discs counting invariants in Definition 4.8.

Definition 4.7. A tropical disc (ϕ, T, w) with stop $u \in B_0$ is called an admissible tropical disc if the following holds:

1. For every vertex $v \in C_0(T)$, its valency $\text{val}(v) \leq 3$.
2. Assume $e \in C_1(T)$ is contracted to a point $\phi(e) \in B$. The preimage of $\phi(e)$ is a disjoint union of subtrees of T . Let T_e be the connected subtree containing e . Let $e_0, e_1, \dots, e_m \in C_1(T)$ be the edges adjacent

to T_e and e_0 is the one closest to the root. Denote the weight of e_i by w_i . Let T' be the tree obtained by adding edges e_0, \dots, e_m with weight w_1, \dots, w_m and \tilde{T} be the tree by replacing each $e_i \in C_1(T')$ by an unbounded edge with weight w_i . $T \setminus T'$ is a disjoint union of subtrees of T . Let T_i be the connected subtree containing e_i , $i = 1, \dots, m$. Then $\phi_i = (\phi|_{T_i}, T_i, w|_{T_i})$ defines a tropical disc with stop at $\phi(e)$. For each e_i there exists a relative class $[\phi_i] \in H_2(X, L_{\phi(e)})$. Let $v_i \in T_{\phi(e)} B_0$ be the primitive vector such that $v_i |Z_{[\phi_i]}| > 0$ and $v_i \text{Arg} Z_{[\phi_i]} = 0$. Let $(v_i, Z_{[\phi_i]}), i = 1, \dots, n$ be such distinct pairs and with the order such that

- (a) $\text{Arg} Z_{[\phi_i]}(\phi(e) - \epsilon v_i) \leq \text{Arg} Z_{[\phi_j]}(\phi(e) - \epsilon v_j)$ for $0 < \epsilon \ll 1$, if $i < j$.
- (b) The above equality holds and $i < j$ imply $|Z_{[\phi_i]}(\phi(e))| < |Z_{[\phi_j]}(\phi(e))|$.

Assume that $w_{ij}, j = 1, \dots, l_i$ are the weights of the edges attached to the pairs $(v_i, Z_{[\phi_i]})$ and ordered in the way that $w_{ij} \leq w_{ij'}$ if $j < j'$. Then there exists $\tilde{\phi} : \tilde{T} \rightarrow \mathbb{R}^2$ with no contracted edges and weight $\tilde{w} : C_1(\tilde{T}) \rightarrow \mathbb{N}$,

$$\tilde{w}(e') = \begin{cases} w(e'), & e \in T' \\ w_i, & e = e_i, \end{cases}$$

such that the balancing condition (2) is satisfied. Moreover, the tropical curve $(\tilde{\phi}, \tilde{T}, \tilde{w})$ is in the standard position with respect to $\{v_i, w_{ij}\}$.

3. We say two admissible tropical discs (ϕ, T, w) and (ϕ', T', w') are equivalent if
 - (a) there exists a homeomorphism $f : T \rightarrow T'$ such that $\phi' \circ f = \phi$ and $w' \circ f = w$, or
 - (b) there exists a subtree T_e of T contracted to a point (with notation above), (ϕ', T', w') is derived from exchanging ϕ_i and ϕ_j , if $[\phi_i] = [\phi_j]$.

Now we will define a weighted count for tropical discs on K3 surfaces following [15], which is motivated from the work of [8].

Definition 4.8. 1. Let $\phi : T \rightarrow B_\vartheta$ be an admissible tropical disc with the stop $u \in B_0$. Then we define its weight of ϕ to be

$$\text{Mult}(\phi) := \prod_{v \in C_0^{\text{int}}(T)} \text{Mult}_v(\phi) \prod_{v \in C_0^{\text{ext}}(T) \setminus \{u\}} \frac{(-1)^{w_v - 1}}{w_v^2} \prod_{T_e : \phi(e) \text{ is a point}} |\text{Aut}(\mathbf{w}_{T(e)})|,$$

where the notation is explained below:

- (a) Here we use the notation in Definition 4.7. Then we set $\mathbf{w}_{T_e} = (\mathbf{w}_1, \dots, \mathbf{w}_n)$. The last product doesn't repeat the factor if $T_e = T_{e'}$.
- (b) For a set of weight vectors $\mathbf{w} = (\mathbf{w}_1, \dots, \mathbf{w}_n)$ and $\mathbf{w}_i = (w_{i1}, \dots, w_{il_i})$, for $i = 1, \dots, n$. We set

$$a_n^i = |\{w_{ij} | w_{ij} = n\}|$$

and

$$|Aut(\mathbf{w})| = \prod_i \prod_{\substack{n \in \mathbb{N} \\ a_n^i \neq 0}} (a_n^i)!,$$

which is the subgroup of the permutation group $\prod_i \Sigma_{l_i}$ stabilizing \mathbf{w} .

- 2. Let $u \in B_0$ and $\gamma \in H_2(X, L_u)$. We define the tropical discs counting invariant $\tilde{\Omega}^{trop}(\gamma; u)$ to be

$$\tilde{\Omega}^{trop}(\gamma; u) := \sum_{\phi} Mult(\phi),$$

where the sum is over all equivalent classes of admissible tropical discs on $B_{ArgZ_\gamma(u)}$ with stop at u such that $[\phi] = \gamma$.

Remark 4.9. 1. The definition of the weighted counting of tropical discs $\tilde{\Omega}^{trop}(\gamma; u)$ does not depend on the choices of the Kähler class $[\omega]$ nor the \mathbb{C}^* -scaling of the holomorphic volume form Ω .

- 2. The central charge can be viewed as the weighted sum of affine length of each edge.

The definition of the tropical discs counting comes from the following reason: Assume that the affine structure with singularities on B comes from the complex affine structure of a special Lagrangian fibration in K3 surface, the hyperKähler rotation will induce an S^1 -family of affine structure with singularities, which we will denote them by $\{B_\vartheta\}$ and $B_{\vartheta=0} = B$ [15].

The following is an analogue of Gromov compactness theorem for tropical discs.

Lemma 4.10. Fix $u \in B_0$ and $\lambda > 0$, there exists finitely many relative $\gamma \in H_2(X, L_u)$ such that $\tilde{\Omega}^{trop}(\gamma; u) \neq 0$ and $|Z_\gamma(u)| < \lambda$.

Proof. First notice that $|Z_\gamma|$ is the sum of affine length of the edges in the image of tropical discs. Consider the intersections of initial rays from each pair of singularities. Let λ_0 be the smallest sum of affine length of initial rays to the intersection point.

Given an admissible tropical disc (ϕ, T, w) , with $|Z_{[\phi]}| < \lambda$, there exists a unique tropical disc (ϕ', T', w') such that ϕ' is injective and ϕ factors through ϕ' . In particular, $Z_{[\phi']} = Z_{[\phi]}$. Then we have

$$(|C_0^{int}(T')| + 1 + |C_0^{ext}(T') - C_0^{int}(T')|)\lambda_0 < |Z_{[\phi']}| < \lambda.$$

Thus, the number of vertices $|C_0(T')|$ is bounded and there are only finitely many such trees. Let w'_i be the weights on the external wedges of T' , then

$$(\sum_i w'_i)\lambda_0 < |Z_{[\phi']}|$$

and there are finitely many possible assignment of weights on external edges of a fixed tree. The weight and the direction of the rest of the internal edges are determined by the balancing conditions.

For a fixed tropical disc (ϕ', T', w') , there are only finitely many admissible tropical discs factoring through it from the definition of admissibility. Actually, we prove a stronger statement that given $\lambda > 0$, there is only finitely many pair (u, γ) , $u \in B_0$ and $\gamma \in H_2(X, L_u)$ such that $\tilde{\Omega}^{trop}(\gamma; u) \neq 0$ and $|Z_\gamma(u)| < \lambda$. \square

4.1 Wall-Crossing Formula and Properties of Tropical Discs Countings

Before we talk about the wall-crossing formula, we will introduce the notion of central charge and wall of marginal stability. First there exists a holomorphic function called central charge

$$\begin{aligned} Z : \Gamma = \cup_{u \in B_0} H_2(X, L_u) &\rightarrow \mathbb{C} \\ \gamma_u &\longmapsto Z_{\gamma_u}(u) := \int_{\gamma_u} \Omega, \end{aligned} \tag{4}$$

which is well-defined since Ω restrict to zero on the fibres.

Definition 4.11. Let $\gamma \in H_2(X, L_{u_0})$, locally we define the locus W_γ^{trop} to be

$$W_\gamma^{trop} = \bigcup_{\substack{\gamma_1 + \gamma_2 = \gamma \\ \langle \gamma_1, \gamma_2 \rangle \neq 0}} W_{\gamma_1, \gamma_2}^{trop},$$

where

$$W_{\gamma_1, \gamma_2}^{trop} = \{u \in B \mid \text{Arg} Z_{\gamma_1} = \text{Arg} Z_{\gamma_2} \text{ and there exists tropical discs of relative class } \gamma_1, \gamma_2 \text{ ends on } u.\}.$$

The locus $W_{\gamma_1, \gamma_2}^{trop}$ is an open closed subset of zero locus of harmonic functions, which is locally union of finitely many smooth curves on B .

Lemma 4.12. *Given a tropical disc (ϕ, T, w) and $e \in C_1(T)$. If $\phi(e)$ is not a point, then $\phi(e)$ is part of a trajectory of a gradient flow*

$$\frac{d}{dt}\phi(t) = \nabla Z_\gamma(\phi(t)),$$

for some relative class γ .

Proof. Since $\phi(e)$ is an affine segment. Locally, it is given by the equation $f_{\gamma_{\bar{e}}} = c$ for some relative class $\gamma_{\bar{e}} \in H_2(X, L)$ such that $\text{Arg} Z_{\gamma_{\bar{e}}}$ is constant along \bar{e} by Definition 4.5. On the other hand, for a given relative class $\gamma \in H_2(X, L)$ the trajectories of the equation

$$\frac{d}{dt}\phi(t) = \nabla Z_\gamma(\phi(t))$$

are also characterized by the same property. Indeed, we have

$$\frac{d}{dt}\text{Arg} Z_\gamma(\phi(t)) = (\nabla F_\gamma)\text{Arg} Z_\gamma = J\nabla F_\gamma(\log |Z_\gamma|) = 0.$$

The second equation follows from the Cauchy-Riemann equation of Z_γ . The third equation holds because for any function f , its gradient ∇f is perpendicular to its level set. So $J\nabla f(f) = 0$. In particular, the tropical disc (ϕ, T, w) can vary smoothly with respect to ϑ if $\phi(G) \cap \Delta = \emptyset$. \square

For a fixed fibre $L_u, u \in B_0$ and $\gamma \in H_2(X, L_u)$ being primitive, We will associate a generating function of the tropical disc of multiple covers of γ given by

$$\log f_\gamma^{trop}(u) = \sum_{d=1}^{\infty} d\tilde{\Omega}^{trop}(d\gamma; u)(T^{Z_\gamma(u)}z^\gamma)^d \in \mathbb{C}[[z^\gamma]] \otimes \Lambda_+^{\mathcal{S}_\gamma(u)}, \quad (5)$$

where $\mathcal{S}_\gamma(u) := \mathbb{R}_{>0}Z_\gamma(u)$ is a ray. The function f_γ^{trop} is known as the slab function in [9]. In particular, we have

$$\begin{aligned} f_\gamma^{trop}(u) &\equiv 1 \pmod{\Lambda_+^{\mathcal{S}_\gamma(u)}} \\ f_\gamma^{trop}(u) &\equiv 1 \pmod{(z^{\partial\gamma})}. \end{aligned}$$

Then given $u \in B_0$, $\lambda > 0$ and a sector \mathcal{S} , we associate a symplectomorphism of the "quantum torus $(\mathbb{C}^*)^2$ " as follows: first we let $\tilde{\theta}_{\gamma, < \lambda}^{trop}(u)$ be an automorphism of algebras through the inclusion $\Lambda^{\mathcal{S}_\gamma}/F^\lambda \Lambda^{\mathcal{S}_\gamma} \subseteq \Lambda^{\mathcal{S}}/F^\lambda \Lambda^{\mathcal{S}}$ and $\mathbb{C}[[z^\gamma]] \subseteq \mathbb{C}[[H_2(X, L_u)]]$,

$$\begin{aligned} \tilde{\theta}_{\gamma, < \lambda}^{trop}(u) : (\Lambda^{\mathcal{S}}/F^\lambda \Lambda^{\mathcal{S}})[[H_2(X, L_u)]] &\rightarrow (\Lambda^{\mathcal{S}}/F^\lambda \Lambda^{\mathcal{S}})[[H_2(X, L_u)]] \\ z^{\gamma'} &\mapsto z^{\gamma'} (f_\gamma^{trop}(u))^{\langle \gamma, \gamma' \rangle} \end{aligned} \quad (6)$$

and

$$\Theta_{\mathcal{S}, < \lambda}^{trop}(u) := \prod_{\substack{\gamma \in H_2(X, L_u) \text{ primitive:} \\ \text{Arg } Z_\gamma(u) \in \mathcal{S}}}^{\curvearrowright} \tilde{\theta}_{\gamma, < \lambda}^{trop}(u). \quad (7)$$

Here the product \prod^{\curvearrowright} is taken in the order of $\text{Arg } Z_\gamma$ and by Lemma 4.10 that (7) is a finite product. The pairing $\langle \gamma, \gamma' \rangle$ is the natural pairing of the boundaries in $H_1(L_u)$. Then the limit

$$\Theta_{\mathcal{S}}^{trop}(u) := \lim_{\lambda \rightarrow \infty} \Theta_{\mathcal{S}, < \lambda}^{trop}(u) \in \text{Aut}(\lim_{\leftarrow} \Lambda^{\mathcal{S}}/(F^\lambda \Lambda^{\mathcal{S}})[H_1(L_u)])$$

exists via Lemma 4.15 and the uniqueness of scattering diagram.

Definition 4.13. *We will call a transformation of the form in (6) an elementary transformation and call f_γ^{trop} the associated slab function.*

It is straight-forward to check that if $\langle \gamma_1, \gamma_2 \rangle = 0$, then $[\tilde{\theta}_{\gamma_1}^{trop}, \tilde{\theta}_{\gamma_2}^{trop}] = 0$. The symplectomorphism $\Theta(u)$ is well-defined in Λ_+ -adic topology due to the above algebra calculation and Lemma 4.10.

Given a path $\phi : [0, 1] \rightarrow B_0$ such that $\phi(0) = u_0$ and $\phi(1) = u_1$, the parallel transport $T_{u_0, u_1} : H_2(X, L_{u_0}) \rightarrow H_2(X, L_{u_1})$ naturally induces a well-defined transformation (which we will still denote by T_{u_0, u_1} and also the induced transformation on the quotients)

$$\begin{aligned} \Lambda[[H_2(X, L_{u_0})]] &\rightarrow \Lambda[[H_2(X, L_{u_1})]] \\ z^{\gamma_{u_0}} &\mapsto T^{Z_\gamma(u_1) - Z_\gamma(u_0)} z^{\gamma_{u_1}}. \end{aligned}$$

We will also use the same notation for the induced action on $\Theta_{\mathcal{S}}^{trop}(u)$. The tropical disc counting invariants $\hat{\Omega}^{trop}(\gamma; u)$ satisfy the Kontsevich-Soibelman wall-crossing formula in the following sense:

Theorem 4.14. *Given $u \in B_0$, $\lambda > 0$ and a sector \mathcal{S} . Assume that there exists no $\gamma \in H_2(X, L_u)$ such that $\tilde{\Omega}'(\gamma; u) \neq 0$, $|Z_\gamma(u)| < \lambda$ and $\text{Arg} Z_\gamma(u) \in \partial \mathcal{S}$. Then there exists a neighborhood $U \ni u$ and $\lambda' = \lambda'(\lambda) > 0$ such that*

$$\Theta_{\mathcal{S}}^{trop}(u_2) = T_{u_1, u_2} \Theta_{\mathcal{S}}^{trop}(u_1) \pmod{F^{\lambda'} \Lambda^{\mathcal{S}}},$$

for generic $u_1, u_2 \in U$. Moreover, we have $\lim_{\lambda \rightarrow \infty} \lambda'(\lambda) = \infty$. In other words, the symplectomorphism $\Theta^{\mathcal{S}}(u)$ is "invariant" under the identification of parallel transport.

First we have the following corollary from [8].

Theorem 4.15. [8][15] *Let $u \in B_0$ and $\lambda > 0$. Let $\{l_{\gamma_i}\}$ the set of affine line (and rays) labeled by γ_i passing through u such that $|Z_{\gamma}(u)| < \lambda$. Let ϕ be an oriented circle around u small enough such that it intersects each of l_{γ_i} exactly twice (and once) and transversally at p_i . Then for any λ ,*

$$\prod_i \left(T_{p_i, u} \tilde{\theta}_{\gamma_i}(p_i) \right)^{\epsilon_i} = Id \pmod{F^{\lambda} \Lambda^{\mathbb{R}_{>0} e^{i\vartheta}}},$$

where the order of the product is with respect to the order of the intersections of l_{γ_i} with ϕ . The exponent ϵ_i is the sign of the pairing between the tangent of the circle and the tangent of the affine line l_{γ_i} (with the direction $|Z_{\gamma_i}|$ is decreasing).

Proof. Let u_+, u_- be on different side of W_{γ}^{trop} with

1. $\text{Arg} Z_{\gamma}(u_+) = \text{Arg} Z_{\gamma}(u_-)$ and $|Z_{\gamma}(u_+)| > |Z_{\gamma}(u_-)|$.
2. there exists no u in the affine line between u_+, u_- (in $X_{\vartheta}, \vartheta = \text{Arg} Z_{\gamma}(u_+)$) with $\gamma' \in H_2(X, L_u)$ such that $u \in W_{\gamma'}^{trop}$ and $|Z_{\gamma'}(u)| < \lambda$.

For each $\lambda > 0$, the above conditions can be achieved by choosing u_+, u_- close enough to W_{γ}^{trop} . Then the difference of the tropical discs counting invariants is given by

$$\tilde{\Omega}^{trop}(d\gamma; u_+) - \tilde{\Omega}^{trop}(d\gamma; u_-) = \sum_{\mathbf{w}: \sum |\mathbf{w}_i| \gamma_i = d\gamma} \frac{N_{\{\partial \gamma_i\}}^{trop}(\mathbf{w})}{|Aut(\mathbf{w})|} \left(\prod_{1 \leq i \leq n, 1 \leq j \leq l_i} \tilde{\Omega}^{trop}(w_{ij} \gamma_i) \right),$$

where $N_{\{\partial \gamma_i\}}^{trop}(\mathbf{w})$ is a counting of tropical rational curves defined in [8] and the theorem follows from Theorem 2.8 [8]. \square

Theorem 4.16. [15] *Assume that $u \notin W_\gamma^{trop}$, then there exist a neighborhood $\mathcal{U} \ni u$ such that tropical discs counting is locally a constant, namely*

$$\tilde{\Omega}^{trop}(\gamma, u) = \tilde{\Omega}^{trop}(\gamma; u'),$$

for every $u' \in \mathcal{U}$.

Proof. First we assume that for all admissible tropical discs (ϕ, T, w) stop at u with $[\phi] = \gamma$ satisfy that $\phi(T) \subseteq B_0$. Then the theorem follows from Lemma 4.12 and the fact that the solution of the first ordinary differential equations has smooth dependence of the initial values. It worth noticing that tropical discs with vertices of valency larger than 3 might not deform smoothly respect to ϑ .

Assume there exists a tropical disc (ϕ, T, w) with respect to ϑ , $[\phi] = \gamma$, and pass through a singularity p . Fix $\lambda > 0$, we may assume that the affine line in phase ϑ from p to u does not intersect any other $W_{\gamma'}$ with $|Z_\gamma| < \lambda$. Thus, $\tilde{\Omega}(\gamma; u) = \tilde{\Omega}(\gamma; u')$ for u' on the affine segment. Choose u_+, u_- near u such that $\text{Arg} Z_\gamma(u_+) > \vartheta$ and $\text{Arg} Z_\gamma(u_-) < \vartheta$. Let γ_e be the relative class of Lefschetz thimble associated to p . Then locally $l_{\pm\gamma_e}$ divide a neighborhood of p in to two regions.

Let l_{γ_i} be the affine lines of phase ϑ , labeled by γ_i , passing through p and $|Z_{\gamma_i}(p)| < |Z_\gamma(p)|$. Choose $0 < \epsilon \ll 1$, and denotes $\{l_i^\pm\}$ be the sets of affine lines with respect to $\vartheta \pm \epsilon$ corresponding to the deformation of $\{l_\gamma\}$ and those affine rays with respect to $\vartheta \pm \epsilon$ labeled with relative classes that admits tropical discs containing some $l_{\gamma_i}^\pm$.

Choose u'_\pm on $l_{\pm\gamma_e}$ close to p such that $-Z_{\gamma_e}(u'_+) = Z_{-\gamma_e}(u'_-)$. Choose counter-clockwise loops ϕ_\pm around u'_\pm such that they intersect all the affine lines (and rays) in $\{l_i^\pm\}$ transversally and exactly twice (and once respectively). We decompose ϕ_\pm into four arcs ϕ_i^\pm , $i = 1, 2, 3, 4$ such that ϕ_1^\pm, ϕ_3^\pm only intersect $l_{\pm\gamma_e}$, ϕ_2^\pm is contained in the region with u and ϕ_4^\pm is contained in the region without u (See Figure 1). Let $F_{\phi_i^\pm}(u'_\pm)$ to be the composition of the transformation (parallel transport to u'_\pm) attached to $l_{\gamma_i}^\pm$ such that ϕ_i^\pm intersect and the composition order is with respect to the order of intersection of $l_{\gamma_i}^\pm$ with ϕ_i^\pm . Then by Theorem 4.15, we have

$$\begin{aligned} F_{\phi_4^+}(u'_+) \circ F_{\phi_3^+}(u'_+) \circ F_{\phi_2^+}(u'_+) \circ F_{\phi_1^+}(u'_+) &= \text{Id} \\ F_{\phi_4^-}(u'_-) \circ F_{\phi_3^-}(u'_-) \circ F_{\phi_2^-}(u'_-) \circ F_{\phi_1^-}(u'_-) &= \text{Id} \end{aligned}$$

Let α denotes the action of (counter-clockwise) monodromy around p ,

$$\alpha(z^{\gamma'}) = z^{\gamma' - \langle \gamma_e, \gamma' \rangle \gamma_e}.$$

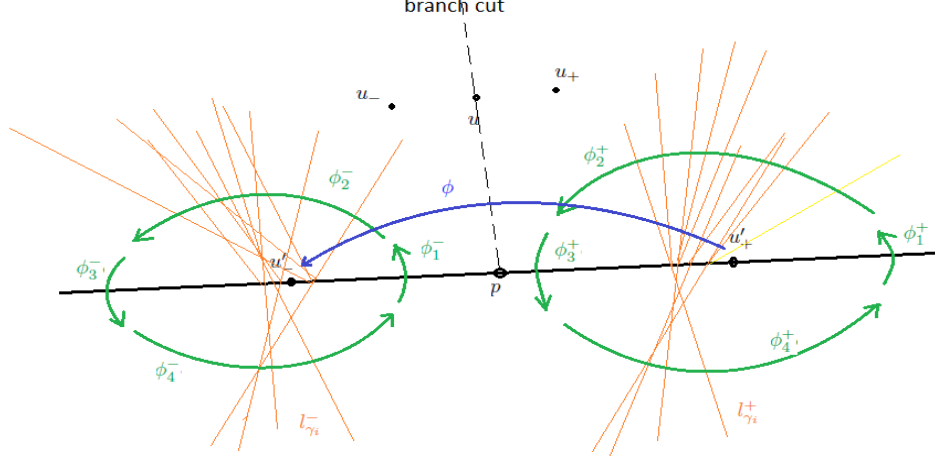


Figure 1: When tropical discs vary across a singularity.

For u' near p , we also define the transformation

$$\theta_{\pm\gamma_e}(u')(z^{\gamma'}) = z^{\gamma'}(1 + z^{\pm\gamma_e}T^{\pm Z_{\gamma_e}(u')})^{\pm 1}.$$

Notice that for any γ' , we have

$$\begin{aligned} \alpha^{-1} \circ (\theta_{-\gamma_e}(u'_+))^{-1}(z^{\gamma'}) &= \alpha^{-1}(z^{\gamma'}(1 + T^{-Z_{\gamma_e}(u'_+)}z^{-\gamma_e})^{\langle\gamma_e, \gamma'\rangle}) \\ &= z^{\gamma' + \langle\gamma_e, \gamma'\rangle\gamma_e}(1 + T^{-Z_{\gamma_e}(u'_+)}z^{-\gamma_e})^{\langle\gamma_e, \gamma'\rangle} \\ &= z^{\gamma'}(1 + T^{Z_{\gamma_e}(u'_+)}z^{\gamma_e})^{\langle\gamma_e, \gamma'\rangle}T^{-Z_{\gamma_e}(u'_+)}^{\langle\gamma_e, \gamma'\rangle}. \end{aligned}$$

Therefore, we have

$$\alpha^{-1} \circ (\theta_{-\gamma_e}(u'_+))^{-1} = T^{-Z_{\gamma_e}(u'_+)}^{\langle\gamma_e, \gamma'\rangle} \theta_{\gamma_e}(u_-). \quad (8)$$

By definition, we have $F_{\phi_3^\pm}(u'_\pm) = \theta_{\pm\gamma_e}(u'_\pm)^{\mp 1}$ and $F_{\phi_1^\pm}(u'_\pm) = \theta_{\pm\gamma_e}(u'_\pm)^{\pm 1}$. From (8), we have

$$\begin{aligned} \text{Id} &= F_{\phi_4^+}(u'_+) \circ \theta_{\gamma_e}(u'_+)^{-1} \circ F_{\phi_2^+}(u'_+) \circ \theta_{\gamma_e}(u'_+) \\ &= F_{\phi_4^+}(u'_+) \circ \theta_{-\gamma_e}(u'_+)^{-1} \circ (\alpha \circ F_{\phi_2^+}(u'_+) \circ \alpha^{-1}) \circ \theta_{\gamma_e}(u'_+)^{-1} \end{aligned} \quad (9)$$

$$(10)$$

Apply $T_{u'_+, u'_-}$ (along a path contained in the region does not contain u) to both sides of (9), we have $F_{\phi_2^-}(u'_-) = T_{u'_+, u'_-} \alpha \circ F_{\phi_2^+}(u_+) \circ \alpha^{-1}$ and we have

$$\tilde{\Omega}(\gamma; u_+) = \tilde{\Omega}(\gamma; u_-).$$

Let $\tilde{\Omega}(\gamma; u_+)_1$ to be the contribution from tropical discs with images which contain p and locally intersect both regions, $\tilde{\Omega}(\gamma; u_+)_2$ to be the contribution from tropical discs with images which do not contain p and locally intersect both regions, $\tilde{\Omega}(\gamma; u_+)_3$ to be the contribution from tropical discs with images locally only intersect the region contains u . Then

$$\tilde{\Omega}(\gamma; u_+) = \tilde{\Omega}(\gamma; u_+)_1 + \tilde{\Omega}(\gamma; u_+)_2 + \tilde{\Omega}(\gamma; u_+)_3.$$

Similarly, we define $\tilde{\Omega}(\gamma; u_-)_1, \tilde{\Omega}(\gamma; u_-)_2$. Then we have

$$\begin{aligned}\tilde{\Omega}(\gamma; u_+)_1 &= \tilde{\Omega}(\gamma; u_-)_2 \\ \tilde{\Omega}(\gamma; u_+)_2 &= \tilde{\Omega}(\gamma; u_-)_1 \\ \tilde{\Omega}(\gamma; u_+)_3 &= \tilde{\Omega}(\gamma; u_-)_3.\end{aligned}$$

In particular, this implies that

$$\tilde{\Omega}(\gamma; u) = \tilde{\Omega}(\gamma; u_+)_1 + \tilde{\Omega}(\gamma; u_-)_1 + \tilde{\Omega}(\gamma; u_+)_3 = \tilde{\Omega}(\gamma; u_+)$$

and this finishes the proof of the theorem. \square

Proof. (of Theorem 4.14) First we assume that $u \notin W_\gamma^{trop}$ for any $\gamma \in H_2(X, L_u)$ with $|Z_\gamma(u)| < \lambda$ and $Z_\gamma(u) \in \mathcal{S}$. Together with the assumption of the Theorem 4.14 and Lemma 4.10, there exists a neighborhood $U \ni u$ such that for every $u' \in U$, we have

1. there exists no $\gamma \in H_2(X, L_{u'})$ with $|Z_\gamma(u')| < \lambda$, $\tilde{\Omega}^{trop}(\gamma; u') \neq 0$ and $\text{Arg} Z_\gamma(u) \in \partial \mathcal{S}$.
2. there exists no $\gamma \in H_2(X, L_{u'})$ with $|Z_\gamma(u')| < \lambda$, $Z_\gamma(u') \in \mathcal{S}$ and $u' \in W_\gamma^{trop}$.

Then the relative class γ with $|Z_\gamma(u')| < \lambda$, $Z_\gamma(u') \in \mathcal{S}$ and $\tilde{\Omega}^{trop}(\gamma; u') \neq 0$ for every $u' \in U$ can be identified via the parallel transport. From Theorem 4.16, we have

$$T_{u_1, u_2} f_\gamma^{trop}(u_1) = f_\gamma^{trop}(u_2) \pmod{T^\lambda} \quad (11)$$

for all $u_1, u_2 \in U$ and γ such that $|Z_\gamma| < \lambda$ on U . Then the theorem follows from Lemma 2.1.

Now assume that $u \in W_{\gamma_1}^{trop} \cap \cdots \cap W_{\gamma_k}^{trop}$ for one $\gamma_i \in H_2(X, L_u)$ with $|Z_\gamma(u)| < \lambda$, $\text{Arg}Z_{\gamma_i}(u) \in \mathcal{S}$. We take $U \ni u$ be a neighborhood such that for any $u' \in U$, we have

1. there is no $\gamma \in H_2(X, L_{u'})$ with $|Z_\gamma(u')| < \lambda$, $\tilde{\Omega}^{trop}(\gamma; u') \neq 0$ and $\text{Arg}Z_\gamma(u') \in \partial\mathcal{S}$.
2. there is no $\gamma \in H_2(X, L_{u'})$ with $|Z_\gamma(u')| = \lambda$, $\tilde{\Omega}^{trop}(\gamma; u') \neq 0$ and $\text{Arg}Z_\gamma(u') \in \mathcal{S}$. (this can be achieved by slightly enlarge λ)
3. there is no $\gamma \in H_2(X, L_{u'})$ with $|Z_\gamma(u')| < \lambda$, $u' \in W_\gamma^{trop} \cap U \neq W_{\gamma_i}^{trop} \cap U$.

Given any $u_1, u_2 \in U$, the relative class γ with $\tilde{\Omega}(\gamma; u_i) \neq 0$, $\text{Arg}Z_\gamma(u_i) \in \mathcal{S}$ and $|Z_\gamma(u_i)| < \lambda$ are identified except those annihilate/creation from the walls $W_{\gamma_i}^{trop}$. It suffices to consider the situation when u_1, u_2 are closed to a single wall $W_{\gamma_i}^{trop}$ and on the different sides (otherwise, it reduces to the previous situation), for some i . Choose a path ϕ connecting u_1, u_2 passing through $W_{\gamma_i}^{trop}$ exactly once at p and do not intersect $W_{\gamma_j}^{trop}$ for $j \neq i$. Let $\{l_{\gamma'_i}\}$ be the sets of affine lines and rays such that γ'_i satisfies $\tilde{\Omega}^{trop}(\gamma'_i; p) \neq 0$ and $\text{Arg}Z_{\gamma'_i}(u) = \text{Arg}Z_{\gamma_i}(p)$. Let ϕ be a small loop around p and falls in U . Then ϕ is separated by $W_{\gamma_i}^{trop}$ into two parts ϕ_1, ϕ_2 . Assume that ϕ_1 is in the same sides of $W_{\gamma_i}^{trop}$ as u_1 . Then there exists a subsector \mathcal{S}_i contains $Z_{\gamma_i}(p)$ in its interior such that

$$\begin{aligned} T_{u_1, p} \left(\prod_{\gamma: \text{Arg}Z_\gamma \in \mathcal{S}'} \tilde{\theta}_{\gamma_i}^{trop}(u_1) \right) &= \prod_{\gamma: l_{\gamma_i} \cap \phi_1 \neq \emptyset} \tilde{\theta}_{\gamma_i}^{trop}(p) \\ &= \prod_{\gamma: l_{\gamma_i} \cap \phi_2 \neq \emptyset} \tilde{\theta}_{\gamma_i}^{trop}(p) = T_{u_2, p} \left(\prod_{\gamma: \text{Arg}Z_\gamma \in \mathcal{S}'} \tilde{\theta}_{\gamma_i}^{trop}(u_2) \right) \pmod{T^\lambda}. \end{aligned} \quad (12)$$

Here the first and third equality comes from the choice of U and the second equality comes from Theorem 4.15. Applying T_{u, u_2} on both sides of (12), we have

$$\Theta_{\mathcal{S}_i}^{trop}(u_2) = T_{u_1, u_2} \Theta_{\mathcal{S}_i}^{trop}(u_1) \pmod{T^\lambda}.$$

For relative classes γ' such that $\tilde{\Omega}^{trop}(\gamma') \neq 0$ and $\text{Arg}Z_{\gamma'} \in \mathcal{S} \setminus \mathcal{S}_i$ along ϕ , they are identified via parallel transport and thus (11) holds for those γ' . The theorem then follows from Lemma 2.1. \square

Remark 4.17. *The Theorem 4.14 is also true under the projection $H_2(X, L_u) \rightarrow H_1(L_u)$.*

Definition 4.18. *A quadratic refinement is an assignment $c : \Gamma \rightarrow \{\pm 1\}$ such that*

$$c(\gamma_1 + \gamma_2) = (-1)^{\langle \partial\gamma_1, \partial\gamma_2 \rangle} c(\gamma_1)c(\gamma_2),$$

for any $\gamma_1, \gamma_2 \in \Gamma$.

We will put proof the existence of quadratic refinement in Appendix A. With the notion of quadratic refinement, the slab function f_γ can be decomposed into unique multiplicative sequence in a unique way as follows:

$$f_\gamma^{trop}(u) = \prod_{d \in \mathbb{N}} (1 - c(d\gamma) z^{d\partial\gamma})^{d\Omega^{trop}(d\gamma; u)} \quad (13)$$

with $\Omega^{trop}(\gamma; u) \in \mathbb{Q}$ a priori. In other words, this introduce a multiplicative decomposition of $\tilde{\theta}_\gamma(u)$ given by

$$\tilde{\theta}_\gamma^{trop}(u) = \prod_d (\theta_\gamma^{trop}(u))^{\Omega(\gamma; u)}.$$

Properties of $\tilde{\Omega}^{trop}(\gamma; u)$ will translate to analogue properties of $\Omega^{trop}(\gamma; u)$. The locally constant $\Omega^{trop}(\gamma; u)$ are known as the generalized Donaldson-Thomas invariants. Then the integrality conjecture [12] states that

Conjecture 4.19. *Under above notation and assumption,*

1. $\Omega^{trop}(\gamma; u) \in \mathbb{Z}$ for $u \notin W_\gamma^{trop}$.
2. $\Omega^{trop}(d\gamma; u) = 0$ for sufficiently large d .

The set of invariants $\{\tilde{\Omega}^{trop}(\gamma; u)\}$ can be derived from the set of invariants $\{\Omega^{trop}(\gamma; u)\}$ recursively and vice versa. Indeed, we have

$$\tilde{\Omega}^{trop}(d\gamma) = - \sum_{k|d} c\left(\frac{d}{k}\gamma\right) d \frac{\Omega^{trop}(\frac{d}{k}\gamma)}{k^2}, \quad (14)$$

for any $d \in \mathbb{Z}$ from equation (5) (13). The equation (14) is also referred as the multiple cover formula for holomorphic discs [6].

Definition 4.20. *The wall of marginal stability for tropical discs are defined by*

$$W_\gamma^{trop} := \{u \in W_\gamma^{ttrop} \mid \Delta \tilde{\Omega}^{trop}(\gamma) \neq 0\}.$$

For $u \in W_\gamma^{ttrop} \setminus W_\gamma^{trop}$, we will define $\tilde{\Omega}^{trop}(\gamma; u)$ by the natural continuous extension.

Remark 4.21. *Similar proofs shows that Theorem 4.16 holds when $u \notin W_\gamma^{trop}$.*

We will leave the proof of the existence of quadratic refinement in the Appendix.

Remark 4.22. *Another way to understand the Kontsevich-Soibelman transformation is to introduce the Kontsevich-Soibelman algebra [12]. Consider the Lie algebra structure on $\Lambda^S / F^\lambda \Lambda^S[[H_1(L_u)]]$ given by*

$$[z^{\partial\gamma_1}, z^{\partial\gamma_2}] = (-1)^{\langle\gamma_1, \gamma_2\rangle} \langle\gamma_1, \gamma_2\rangle z^{\partial\gamma_1 + \partial\gamma_2}.$$

Then the transformation θ_γ admits another expression given by

$$\theta_\gamma(u) = \exp ad(Li_2(T^{Z_\gamma(u)} z^{\partial\gamma})), \quad (15)$$

where Li_2 is the dilogarithm function $Li_2(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^2}$. More generally, an elementary transformation

$$z^{\partial\gamma'} \mapsto z^{\partial\gamma'} f^{\langle\partial\gamma, \partial\gamma'\rangle}$$

for $f \in 1 + \Lambda[[z^{\partial\gamma}]]$ can be expressed of product of term in the form (15).

4.2 Refined Tropical Countings

The Kontsevich-Soibelman algebra can be q -deformed as follows [12]: the algebra becomes non-commutative

$$z^{\partial\gamma_1} z^{\partial\gamma_2} = q^{\langle\gamma_1, \gamma_2\rangle} z^{\partial\gamma_1} z^{\partial\gamma_2},$$

where q is a formal variable. The Lie bracket is replaced by

$$[z^{\partial\gamma_1}, z^{\partial\gamma_2}] = (q^{n/2} - q^{-n/2}) z^{\partial\gamma_1 + \partial\gamma_2}.$$

The q -deformed dilogarithm

$$Li_2(z; q) := \sum_{k=1}^{\infty} \frac{z^k}{k(1-q^k)}$$

induces a q -deformation of θ_γ by

$$\theta_{q,\gamma}(u) = Ad(Li_2(-q^{1/2}T^{Z_\gamma(u)}z^{\partial\gamma})).$$

With the above notation, we are ready for the definition of refined tropical discs counting invariants.

Definition 4.23. 1. Let $\phi : T \rightarrow B$ be an admissible tropical disc with the stop $u \in B_0$ with the image $\bar{T} = \phi(T)$ be the immersed graph on B . Let $C_0(\bar{T})$ (and $C_1(\bar{T})$) denotes the set of vertices (and the edges respectively). Then we define its weight of ϕ to be

$$Mult_q(\phi) := \prod_{v \in C_0^{int}(T)} [Mult_v(\phi)]_q \prod_{v \in C_0^{ext}(T) \setminus \{u\}} \frac{(-1)^{w_v-1}}{w_v[w_v]_q} \prod_{\bar{v} \in C_0^{int}(\bar{T})} |Aut(\mathbf{w}_{\bar{v}})|.$$

Here we use the notation

$$[n]_q := \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}, \text{ for } n \in \mathbb{N}$$

and the notations are the same in Definition 4.8.

2. Let $u \in B_0$ and $\gamma \in H_2(X, L_u)$. The refined tropical discs counting invariant $\tilde{\Omega}_q^{trop}(\gamma; u)$ is defined to be

$$\tilde{\Omega}_q^{trop}(\gamma; u) := \sum_{\phi} Mult_q(\phi),$$

where the sum is over all admissible tropical discs on B with stop at u such that $[\phi] = \gamma$.

The definition of refined tropical discs counting again looks artificial. However, they satisfy the refined wall-crossing formula.

Theorem 4.24. Then the refined tropical discs counting invariants satisfy the refined wall-crossing formula.

Proof. The proof is essentially the same as the proof of Theorem 4.14. The refined analogue of Theorem 4.15 is given by Corollary 4.9 in [7]. \square

Together with the Theorem 6.28 which states that

$$\tilde{\Omega}(\gamma; u) = \tilde{\Omega}^{trop}(\gamma; u),$$

this suggests that the open Gromov-Witten invariants $\tilde{\Omega}(\gamma; u)$ also admits an refinement. We will leave the geometric interpretation of the refined open Gromov-Witten invariants for future work.

5 Lagrangian Floer Theory of Special Lagrangians in hyperKähler Manifolds

This section we will review some results in Lagrangian Floer theory and its application to special Lagrangians in hyperKähler manifolds. In particular, we will use Λ to be the standard Novikov ring

$$\Lambda := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}, \text{ and } \lambda_i \text{ increasing, } \lim \lambda_i = +\infty \right\}.$$

and Λ^+ be the subset of Λ with positive valuation.

Let (X, ω) be a symplectic manifold with a compatible almost complex structure J . Let L be a Lagrangian submanifold in X . From the real codimension one boundary of the moduli space of holomorphic discs, Fukaya proved

Theorem 5.1. [3] *There exists an gapped filtered A_∞ structure $\{m_{k,\gamma}\}_{k \geq 0, \gamma \in H_2(X, L)}$ on the de Rham complex $\Omega^*(L, \Lambda)$ unique up to pseudo-isotopies.*

From Theorem 5.1 and Theorem 2.5, we have the following corollary:

Corollary 5.2. [3] *There exists an gapped filtered A_∞ structure on $H^*(L, \Lambda)$ unique up to pseudo-isotopies.*

A priori, the A_∞ structure in Corollary 5.2 depends on the choices of the almost complex structures. For any two compatible almost complex structures J, J' , one follows Theorem 5.1 and constructs two A_∞ structures $\{m_{k,\gamma}\}$ and $\{m'_{k,\gamma}\}$ on $\Omega^*(L, \Lambda)$. Moreover, one can find a 1-parameter family of compatible almost complex structure $\{J_t\}_{t \in [0,1]}$ connecting J, J' . Fukaya proved

Theorem 5.3. [3] *There exists an pseudo-isotopy $(\{m_{k,\gamma}^t\}, \{c_{k,\gamma}^t\})$ between the two A_∞ structures induced by the almost complex structure J and J' .*

Thus the A_∞ structure on $H^*(L, \Lambda)$ can be viewed as an symplectic invariant and does not depend on the choices of almost complex structure up to pseudo-isotopies. By constructing compatible forgetful maps, we have the analogue of divisor axiom in Gromov-Witten theory:

Theorem 5.4. [4] *Let $k \geq 0$ and $b, x_0, \dots, x_k \in H^1(L, \Lambda)$, then*

$$\begin{aligned} \sum_{m_0 + \dots + m_k = m} m_{k+m, \gamma}(b^{\otimes m_0}, x_1, b^{\otimes m_1}, \dots, b^{\otimes m_{k-1}}, x_k, b^{\otimes m_k}) \\ = \frac{1}{m!} \langle \partial\gamma, b \rangle^m m_k(x_1, \dots, x_k). \end{aligned}$$

Assume that $\mathcal{M}_\gamma(X_\vartheta, L_{u_0})$ is non-empty and L_u is a nearby smooth special Lagrangian with the same property. Write $\partial\gamma = \sum_i a_i e_i$, for some $a_i \in \mathbb{Z}$, then

$$\sum_i a_i (f_i(u) - f_i(u_0)) = a_i \left(\int_{\gamma_u} \text{Im}\Omega - \int_{\gamma_{u_0}} \text{Im}\Omega \right) = 0 - 0 = 0.$$

Thus, we reach the following proposition:

Proposition 5.5. *Assume that L_u is a nearby smooth special Lagrangian such that $\mathcal{M}_\gamma(X, L_{u_0})$ and $\mathcal{M}_\gamma(X, L_u)$ are non-empty for some relative class γ , then u falls on an affine hyperplane passing through u_0 .*

We will say an affine line l in Proposition 5.5 is labeled by γ (locally) and denoted it by l_γ . Since $H_2(X, L_u)$ is countable, we have the following corollary:

Corollary 5.6. *For any generic point $u \in B_0$, the special Lagrangian submanifold L_u bounds no holomorphic discs.*

The Proposition 5.5 together with Gromov compactness theorem gives the follow corollary:

Corollary 5.7. *Fix $\lambda > 0$, then the set*

$$L_\lambda := \{u \in B_0 \mid L_u \text{ bounds a holomorphic discs with area less than } \lambda. \}$$

is locally a closed subset of finite union of walls of first kind with energy less than λ .

The central charge function (4) plays an important roles for holomorphic discs with special Lagrangian boundary condition in hyperKähler manifolds.

Lemma 5.8. [15] *If $\mathcal{M}_\gamma(\mathfrak{X}^{[\omega]}, L) \neq \emptyset$, then $\mathcal{M}_\gamma(\mathfrak{X}^{[\omega]}, L) = \mathcal{M}_\gamma(X_\vartheta, L)$, $\vartheta = \text{Arg}Z_\gamma$ as topological spaces.*

Assume $\gamma \in H_2(X, L_u)$ and $\mathcal{M}_\gamma(X, L_u)$ has non-trivial real codimension one boundary. Namely, there exists $\gamma_1, \gamma_2 \in H_2(X, L_u)$ such that $\gamma = \gamma_1 + \gamma_2$ and u satisfies the equation

$$\text{Arg}\left(\int_{\gamma_1} \omega + i\text{Im}\Omega\right) = \text{Arg}\left(\int_{\gamma_2} \omega + i\text{Im}\Omega\right) = 0,$$

which cuts out finitely many points on B_0 [15].

For our purpose, we need the following Fukaya's trick: Given a reference point $p \in B_0$ and a path ϕ in a neighborhood U of p such that $\phi(0) = u_0, \phi(1) = u_1$. Assume that U is small enough so that there exists a 2-parameter family of fibrewise preserving diffeomorphisms $\psi_{s,t}$ of the K3 surface satisfying

1. $\psi_{0,t} = \text{Id}$ and $\psi_{1,t}(L_{\phi(t)}) = L_p$. Thus, $\psi_{s,t}(L_{\phi(t)})$ induces a path from $\phi(t)$ to p for fixed t .
2. Let J be the almost complex structure of the K3 surface X . The complex structure $(\psi_{s,t})_*J$ is tame with respect to the symplectic form ω .
3. The diffeomorphism $\psi_{s,t}$ is an identity away from L_u , where u is outside of the neighborhood U .

Then for each relative class $\gamma \in H_2(X, L_p)$, the two moduli spaces of holomorphic discs ¹⁰

$$\mathcal{M}_\gamma(X, L_{\phi(t)}) \rightarrow \mathcal{M}_\gamma(X, L_p) \tag{16}$$

$$\alpha \mapsto (\psi_{1,t})_* \circ \alpha \tag{17}$$

are naturally identified. Here the later moduli space is with respect to the complex structure J , while the former one is respect to the complex structure $(\psi_{1,t})_*J$. Thus via the choice of 2-parameter family of diffeomorphisms $\phi_{s,t}$, we can have a 1-parameter family of (almost) complex structures $\{J_t = (\psi_{1,t})_*J\}_{t \in [0,1]}$. Write the symplectic affine coordinate of u associates to the

¹⁰Here there is a natural identification $H_2(X, L_{u_0})$ between $H_2(X, L_{\phi(t)})$ via $\psi_{1,t,*}$, which is the parallel transport of the Jacobian fibration.

basis is u_1, \dots, u_{2g} , where $u_i = \int_{\bar{e}_i} \omega$. Since J_t is a small perturbation of complex structure of J , Corollary 5.2 gives a pseudo-isotopy between

$$(\Omega^*(L_{u_0}, \Lambda), \sum_{\gamma} m_{k,\gamma} T^{\int_{\gamma} \omega}) \text{ and } (\Omega^*(L_{u_1}, \Lambda), m_{k,\gamma} T^{\int_{\gamma} \omega - \sum u_i(\partial\gamma, e_i)}),$$

where $\partial\gamma = \sum_k (\partial\gamma, e_i) e_i$. In particular, it induces an isomorphism on the Maurer-Cartan spaces of the canonical models ¹¹,

$$F_{(\phi,p)}^{can} : H^1(L_{u_0}, \Lambda_+) = \mathcal{MC}(H^*(L_{u_0}, \Lambda_+)) \cong \mathcal{MC}(H^*(L_{u_1}, \Lambda_+)) = H^1(L_{u_1}, \Lambda_+).$$

First of all, the isomorphism $F_{(\phi,p)}^{can}$ is independent of the choices made to construct the Kuranishi structures and perturbed multi-sections. Indeed, the different choices will induces pseudo-isotopies of the pseudo-isotopies and $F_{(\phi,p)}^{can}$ is well-defined by Theorem 2.12.

For a different choices of the 2-parameter family of the diffeomorphisms $\psi_{s,t}$ and $\psi'_{s,t}$, the resulting isomorphism $F_{(\phi,p)}^{can}$ and $(F_{(\phi,p)}^{can})'$ on the Maurer-Cartan spaces are the same due to the natural isomorphism (16). Moreover, for a different choice of reference point p' , the resulting isomorphisms on the Maurer-Cartan spaces are related by

$$F_{(\phi,p)}^{can} = T_{p',p} F_{(\phi,p')}^{can}.$$

The following proposition follows directly from the construction of F_{ϕ}^{can} .

Proposition 5.9. *Fix $\lambda > 0$. Assume that $L_{\phi(t)}, t \in [0, 1]$ does not bound any holomorphic discs of relative γ such that $|Z_{\gamma}(p)| < \lambda$, then the isomorphism*

$$F_{(\phi,p)}^{can} = Id_{H^1(L, \Lambda_+)} \pmod{T^{\lambda}}.$$

Lemma 5.10. *For any $b_1, b_2 \in H^1(L_u)$ and $b_1 \neq b_2$, then b_1 is not gauge equivalent to b_2 .*

Proof. The lemma follows from Corollary 5.6 when L_u is a generic torus fibre. Together with Proposition 2.12 and Fukaya's trick prove the lemma for general case. \square

Similar argument to Theorem 5.4, we have the divisor axiom for both $m_{k,\gamma}^t$ and $c_{k,\gamma}^t$.

¹¹Notice that the A_{∞} structure of the later one is slightly changed

Theorem 5.11. *Given a pseudo-isotopy $(m_{k,\gamma}^t, c_{k,\gamma}^t)$ between A_∞ -structures, then similar statement of Theorem 5.4 holds for $m_{k,\gamma}^t$ and $c_{k,\gamma}^t$.*

The following is the key theorem to prove the wall-crossing formula of open Gromov-Witten invariants in the later section.

Theorem 5.12. [3] [26] *Let ϕ_0, ϕ_1 be two paths with same end points and homotopic to each other, within a small enough open neighborhood $U \ni p$ in B_0 . Then $F_{(\phi_0,p)}^{can} = F_{(\phi_1,p)}^{can}$. In particular, if ϕ is a loop contractible in a small enough open set $U \subseteq B_0$. Then $F_{(\phi,p)}^{can} = Id$.*

Proof. Assume that $\Phi = \Phi(s, t)$ is a homotopy between ϕ_0 and ϕ_1 , i.e.,

$$\begin{aligned}\Phi(0, t) &= \phi_0(t) \\ \Phi(1, t) &= \phi_1(t).\end{aligned}$$

Then the homotopy Φ induces a pseudo-isotopy of pseudo-isotopy. Then the theorem follows from Theorem 2.13. \square

6 Open Gromov-Witten Invariants on K3 Surfaces

6.1 Open Gromov-Witten Invariants

In this section, we will restrict ourselves to elliptic K3 surfaces¹². Given an elliptic K3 surface $X \rightarrow B$ and a Kähler class $[\omega]$, there exists an S^1 -family of hyperKähler structures such that the fibration $X \rightarrow B$ become special Lagrangian fibration. We will denote the corresponding hyperKähler manifold by $X_\vartheta \in S^1$.

Similar to the story of tropical discs, we have an analogue definition for holomorphic discs:

Definition 6.1. *Let γ be a relative class, locally we define the locus W'_γ to be*

$$W'_\gamma = \bigcup_{\substack{\gamma_1 + \gamma_2 = \gamma \\ \partial\gamma_1 \text{ is not a non-zero multiple of } \partial\gamma_2}} W'_{\gamma_1, \gamma_2},$$

where

$$W'_{\gamma_1, \gamma_2} = \{u \in B \mid \text{Arg} Z_{\gamma_1} = \text{Arg} Z_{\gamma_2} \text{ and there exists holomorphic discs of relative class } \gamma_1, \gamma_2 \text{ ends on } L\}$$

¹²See [18] for the open Gromov-Witten invariants with rigid special Lagrangian boundary conditions.

Then locally W'_γ is a closed subset of the real codimension one locus cut out by

$$\text{Arg}Z_{\gamma_1} = \text{Arg}Z_{\gamma_2}.$$

Now assume $\gamma \in H_2(X, L_u)$ is primitive such that $u \neq W'_\gamma$, then the moduli space

$$\mathcal{M}_\gamma(\mathfrak{X}, L_u) = \cup_{\vartheta \in S^1} \mathcal{M}_\gamma(X_\vartheta, L_u)$$

is compact without boundary (thus no real codimension one boundary)[15]. In particular, the moduli space admits a virtual fundamental class $[\mathcal{M}_\gamma(\mathfrak{X}, L_u)]^{vir}[6][15]$. We may define the open Gromov-Witten invariants as follows:

$$\tilde{\Omega}(\gamma; u) := \int_{[\mathcal{M}_\gamma(\mathfrak{X}, L_u)]^{vir}} 1.$$

In general, we define the open Gromov-Witten invariants via the smooth correspondence in [15],

$$\tilde{\Omega}(\gamma; u) := \text{Corr}_*(\mathcal{M}_\gamma(\mathfrak{X}, L_u); tri, tri)(1).$$

We will refer the readers to [3] for the definition and details about the smooth correspondences. The following are some properties of the open Gromov-Witten invariants $\tilde{\Omega}(\gamma; u)$.

Proposition 6.2. [15] *Assume that $u \notin W'_\gamma$ then $\tilde{\Omega}(\gamma; u)$ is well-defined. Moreover, we have the following properties:*

1. *The invariants $\tilde{\Omega}(\gamma; u)$ is independent of choice of the Kähler class $[\omega]$.*
2. *If there exists a path connecting $u, u' \in B_0$ which does not pass through W'_γ , then*

$$\tilde{\Omega}(\gamma; u) = \tilde{\Omega}(\gamma; u').$$

In particular, W'_γ locally divide B_0 into chambers and $\tilde{\Omega}(\gamma; u)$ is locally constant in each chamber.

3. *Let $u \in W'_\gamma$ generic and let u_+, u_- be on the different side of W'_γ near u . Assume that there exists no γ_i such that $\sum_i \gamma_i = \gamma$, $\text{Arg}Z_{\gamma_i} = \text{Arg}Z_\gamma$ and $\tilde{\Omega}(\gamma; u) \neq 0$, then*

$$\tilde{\Omega}(\gamma; u_+) = \tilde{\Omega}(\gamma; u_-).$$

4. (Reality condition) $\tilde{\Omega}(\gamma; u) = \tilde{\Omega}(-\gamma; u)$.

The open Gromov-Witten invariants near an I_1 -type singular fibre is calculated via Theorem 6.8 and a cobordism argument.

Theorem 6.3. [15] *Let L_{u_0} be an type I_1 -singular fibre. Then there exists a sequence neighborhood U_d of u_0 , $U_{d+1} \subseteq U_d$ such that for each $u \in U_d$,*

$$\tilde{\Omega}(\gamma; u) = \begin{cases} \frac{(-1)^{d-1}}{d^2}, & \gamma = d\gamma_e \\ 0, & \text{otherwise,} \end{cases}$$

where γ_e is the relative class of the Lefschetz thimble associate to L_{u_0} .

One of the application of the open Gromov-Witten invariants is the weak correspondence theorem, which gives a sufficient condition for the existence of tropical discs.

Theorem 6.4. [15] *Let $u \in B_0$ and $\gamma \in H_2(X, L_u)$ such that $u \notin W'_\gamma$ and $\tilde{\Omega}(\gamma; u) \neq 0$. Then there exists a tropical disc (ϕ, T, w) such that $[\phi] = \gamma$.*

6.2 Local Model: Focus-Focus Singularity

Definition 6.5. *The Ooguri-Vafa space X_{OV} is an elliptic fibration over a unit disc with a unique singular fibre, a single node rational curve, over the origin.*

From the explicit coordinate description, there exists a natural S^1 -action which preserves the complex structure. There exists a hyperKähler metric on X_{OV} , realized as a periodic Gibbons-Hawking ansatz, thus is S^1 -invariant. Conversely, any S^1 -invariant hyperKähler metric on the same underlying space is in this form [10].

For a fixed $\vartheta \in S^1$, the complex affine coordinate associated to the special Lagrangian fibration in $(X_{OV})_\vartheta$ gives rise to an affine structure on B with a singularity at the origin. The monodromy of the affine structure around the origin is conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. There are two affine rays l_\pm emanate from the origin in the monodromy invariant direction. Chan studied the holomorphic discs in the Ooguri-Vafa space using the maximal principle and the S^1 -invariance of the hyperKähler structure.

Proposition 6.6. [2]

1. *There exist a simple holomorphic disc with boundary in each special Lagrangian torus fibre over l_\pm . Moreover, the image of the holomorphic disc is smooth.*

2. The above holomorphic discs together with their multiple cover are the only holomorphic discs in $(X_{OV})_{\vartheta}$.

Remark 6.7. For each $u \in l_{\pm}$, the affine segment (with multiplicity) from the origin to u gives a tropical disc with stop at u which corresponds to the simple holomorphic disc (or its multiple cover) in Proposition 6.6.

As the phase ϑ goes around S^1 , the two affine rays will also rotate around the origin counterclockwise and every point will be exactly swept once. In other words, every torus fibre bounds exactly a simple disc (up to orientation) which is holomorphic to some complex structure $(X_{OV})_{\pm\vartheta}$.

The holomorphic disc in Proposition 6.6 falls in a small neighborhood U_{OV} of the singular point of the singular fibre. One can find a 1-parameter family of S^1 -invariant hyperKähler structures connecting the one on U_{OV} and a open set of $T\mathbb{P}^1$ with Eguchi-Hanson metric. The later admits an anti-symplectic involution and every holomorphic disc can be doubled to a real rational curve. One than can use the localization to compute the open Gromov-Witten invariants.

Theorem 6.8. [15] Let $X = X_{OV}$ be the Ooguri-Vafa space and γ_e to be the relative class of the Lefschetz thimble. Then the open Gromov-Witten invariants on X is calculated:

$$\tilde{\Omega}(\gamma; u) = \begin{cases} \frac{(-1)^{d-1}}{d^2}, & \text{if } \gamma = d\gamma_e, d \in \mathbb{Z}. \\ 0, & \text{otherwise.} \end{cases}$$

Let $p_{\pm} \in l_{\pm}$ and U_{\pm} is a small neighborhood of p_{\pm} . Let $u_1^{\pm}, u_2^{\pm} \in U_{\pm}$ be on the different side of l_{\pm} as in Figure 2. Choose a path $\phi_{\pm} : [0, 1] \rightarrow B_0$ such that $\phi_{\pm}(0) = u_1^{\pm}$, $\phi_{\pm}(1) = u_2^{\pm}$ and intersect l_{\pm} exactly once and transversally. The family of Lagrangian induces a pseudo-isotopy between the A_{∞} structures on $H^*(L_{u_1})$ and $H^*(L_{u_2})$. In particular, the pseudo-isotopy induces an isomorphism $F_{(\phi_{\pm}, p)}^{can}$ between $\mathcal{MC}(L_{u_1})$ and $\mathcal{MC}(L_{u_2})$. The explicit expression of the isomorphism follows directly from Theorem 6.8, Theorem 6.25 and equation (24).

Theorem 6.9. Let e_1, e_2 be an symplectic integral basis of $H^1(L_{u_i}, \mathbb{Z})$ and e_1 is the Poincaré dual of the vanishing cycle γ_e .

$$\begin{aligned} F_{(\phi_{\pm}, p_{\pm})}^{can}(z_1) &= z_1 \\ F_{(\phi_{\pm}, p_{\pm})}^{can}(z_2) &= z_2(1 + T^{\pm Z_{\gamma_e}(p_{\pm})} z_1^{\pm 1})^{\pm 1}, \end{aligned} \tag{18}$$

where γ_e is the relative class of Lefschetz thimble.

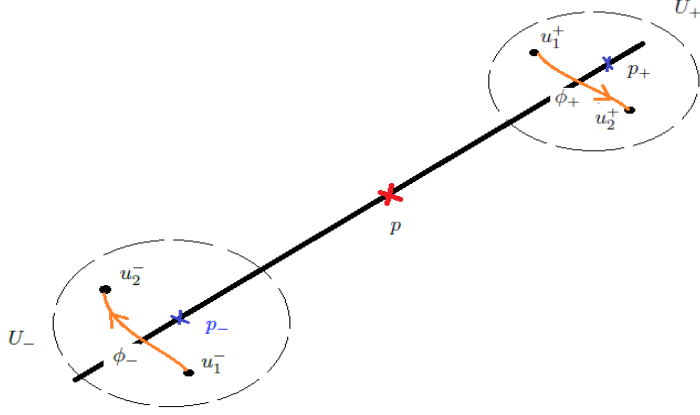


Figure 2: On the base of Ooguri-Vafa space

6.3 Wall-Crossing of Maurer-Cartan Elements

Now we want to understand the wall-crossing phenomenon of the open Gromov-Witten invariants. Given a basis $e_1, e_2 \in H_1(L, \mathbb{Z})$ such that $\langle e_1, e_2 \rangle = 1$, there exists a natural symplectic 2-form on $H^1(L_u; \Lambda_+)$ given by

$$\varpi = \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}.$$

Lemma 6.10. *The symplectic 2-form ϖ is independent of choice of the basis $e_i \in H_1(L, \mathbb{Z})$.*

Proof. Since any two basis are related by a transform $A = GL(2, \mathbb{Z})$, which can be decomposed into product of elementary transformation. It is straightforward to check that ϖ is the same if the two basis is differed by an elementary transformation. \square

Now assume that X is K3 surface and B is the parametrizes the special Lagrangian surfaces with isolated singularity with genus g . Let $p \in B_0$ and $U \subseteq B_0$ is a small enough neighborhood of p . Let $u_0, u_1 \in U$. Let ϕ be a path in U connecting u_0, u_1 such that "Fukaya's trick" applies (see Section 5). We will use the same notation e_1, e_2 for their parallel transport along ϕ .

Write $b = x_1 e_1 + x_2 e_2 \in H^1(L_u, \Lambda_+)$. Using equation (140) of [3], we can compute the form of $F_{(\phi,p)}^{can}$ with respect the above coordinate:

$$\begin{aligned} F_{(\phi,p)}^{can} : H^1(L_{u_0}, \Lambda_+) &\longrightarrow H^1(L_{u_1}, \Lambda_+) \\ b = x_1 e_1 + x_2 e_2 &\mapsto (F_{(\phi,p)}^{can}(b))_1 e_1 + (F_{(\phi,p)}^{can}(b))_2 e_2 \end{aligned}$$

and gives a transformation $x_k \mapsto (F_{(\phi,p)}^{can}(b))_k$, $k = 1, 2$. Motivated by mirror symmetry, it is more natural to consider $z_k = \exp(x_k)$ and transformation $F_{(\phi,p)}^{can}$ acts on z_k by $z_k \mapsto \exp(F_{(\phi,p)}^{can}(b))_k$. This defines an automorphism of the Tate algebra [26]. It is straight-forward to check that the transformation induced by $F_{(\phi,p)}^{can}$ is independent of the choice of the integral basis $e_1, e_2 \in H^1(L, \mathbb{Z})$.

Let $\mathcal{M}_{k,\gamma}^t$ be the moduli spaces of stable discs (holomorphic with respect to J_t) with boundary on L_p and k boundary marked points. We will denote the family moduli space by $\tilde{\mathcal{M}}_{k,\gamma} = \cup_{t \in [0,1]} \mathcal{M}_{k,\gamma}^t$. Let tri denotes the trivial map $\tilde{\mathcal{M}}_{k,\gamma} \rightarrow pt$ to a point. Let ev_k be the evaluation map at the k -th boundary marked point and ev_t be the evaluation map to $[0, 1]$. Recall that the $c_{0,\gamma}^t$ of the pseudo-isotopy in Theorem 5.3 is constructed via smooth correspondences

$$Corr_*(\tilde{\mathcal{M}}_{1,\gamma}; tri, (ev_0, ev_t))(1) = m_{0,\gamma}^t + dt \wedge c_{0,\gamma}^t.$$

Fukaya further considers

$$Corr_*(\tilde{\mathcal{M}}_{0,\gamma}; tri, ev_t)(1) = m_{-1,\gamma}^t + c_{-1,\gamma}^t dt$$

for understanding the wall-crossing phenomenon of the holomorphic discs [3]. With the above notation, we have the following lemma to help understand further the form of $F_{(\phi,p)}^{can}$:

Lemma 6.11. *Let $b \in \Omega^1(L_p)$ be a closed 1-form and γ be a relative class, then*

$$\int_{L \times [t_0, t_1]} c_{0,\gamma}^t \wedge b dt = \int_{[t_0, t_1]} c_{-1,\gamma}^t dt \int_{\partial\gamma} b, \quad (19)$$

for any $[t_0, t_1] \subseteq [0, 1]$.

Proof. Let forget denotes the forgetful map $\tilde{\mathcal{M}}_{1,\gamma} \rightarrow \tilde{\mathcal{M}}_{0,\gamma}$. In terms of

smooth correspondence, the left hand side of (6.11) is

$$\begin{aligned}
& \int_{L \times [t_0, t_1]} \text{Corr}_*(\tilde{\mathcal{M}}_{1, \gamma}; \text{tri}, (ev_0, ev_t))(1) \wedge b \\
&= \text{Corr}_*(\tilde{\mathcal{M}}_{1, \gamma}; (ev_0, ev_t), \text{tri})(b) \\
&= \text{Corr}_*(\tilde{\mathcal{M}}_{1, \gamma}; (ev_0, ev_t), \text{tri} \circ \text{forget})(b) \\
&= \text{Corr}_*(\tilde{\mathcal{M}}_{0, \gamma}; \text{tri}, \text{tri})(1) \cdot \int_{\gamma} b \\
&= \int_{[t_0, t_1]} c_{-1, \gamma}^t dt \int_{\partial \gamma} b.
\end{aligned}$$

The first equality is the composition formula of smooth correspondences (Lemma 4.3[3]). The third equality comes from integration along the fibre of the forgetful map and the compatibility of the forgetful map [3]. \square

Lemma 6.11 implies that for any smooth function b' on L_p , we have

$$\int_{L_p} b'(dc_{0, \gamma}^t) = - \int_{L_p} c_{0, \gamma}^t \wedge db' = 0.$$

In particular, we have

Corollary 6.12. $c_{0, \gamma}^t$ is d -closed 1-form and $\langle \partial \gamma, c_{0, \gamma}^t \rangle = 0$

Now we can state the key theorem to compute the open Gromov-Witten invariants $\tilde{\Omega}'(\gamma; u)$ defined in Section 6.5.

Assume that the elliptic K3 X is generic in the sense that $\Omega : H_2(X, \mathbb{Z}) \rightarrow \mathbb{C}$ is injective. So that $l_\gamma \neq l_{\gamma'}$ unless $\gamma = k\gamma'$ for some $k \in \mathbb{Q}$. Let u_\pm be two points on the different side of l_γ and ϕ is a path connecting u_\pm . We may take u_\pm be close enough to l_γ such there is only relative class $d\gamma$, where $d \in \mathbb{N}$ and γ primitive, can be realized as holomorphic discs with boundary on $L_{\phi(t)}$ for some t and with symplectic area less than λ for a given $\lambda > 0$.

Theorem 6.13. The transformation $F_{(\phi, p)}^{\text{can}}$ is of the form

$$z^{\partial \gamma'} \mapsto z^{\partial \gamma'} f_{\gamma}^{\langle \gamma', \gamma \rangle} \pmod{T^\lambda},$$

for some $f_\gamma \in 1 + \Lambda[[z^{\partial \gamma}]]$. Here we use the notation that

$$z^{\partial \gamma} = z_1^{\langle \gamma, e_1 \rangle} z_2^{\langle \gamma, e_2 \rangle}.$$

Proof. By Proposition 2.12, we have

$$F_{(\phi,p)}^{can}(b) = \sum_{k \geq 1, d \geq 1} \mathfrak{c}(k, d\gamma)(b, \dots, b) T^{d\omega(\gamma)}, \quad (20)$$

where $\mathfrak{c}(k, \gamma)$ is defined in Theorem 2.9. Let $\mathfrak{c}'(k, d\gamma)(b)$ denotes the sum of the terms in (20) which are corresponding to trees with k internal vertices. First we will show that

$$\mathfrak{c}'(k, d\gamma)(b) = 0 \quad (21)$$

(before integrating over the time allocation) by induction on k when $k \geq 2$. For $k = 2$, we have

$$\mathfrak{c}'(2, d\gamma)(b) = \sum_{l_1, l_2 \geq 0} \sum_{i=1}^{l_1-1} \int_{\tau_1 > \tau_2} c_{l_1, \gamma}^{\tau_1}(\underbrace{b, \dots, b}_{i \text{ copies}}, c_{l_2, \gamma}^{\tau_2}(b, \dots, b), \dots, b).$$

By divisor axiom, we have

$$\begin{aligned} & \sum_{i=1}^{l_1-1} c_{l_1, \gamma}^{\tau_1}(\underbrace{b, \dots, b}_{i \text{ copies}}, c_{l_2, \gamma}^{\tau_2}(b, \dots, b), \dots, b) \\ &= \frac{\langle \partial\gamma, b \rangle^{l_1-1}}{(l_1-1)!} c_{1, \gamma}^{\tau_1}(c_{l_2, \gamma}^{\tau_2}(b, \dots, b)) \\ &= \frac{\langle \partial\gamma, b \rangle^{l_1+l_2-1}}{(l_1+l_2-1)!} c_{1, \gamma}^{\tau_1}(c_{0, \gamma}^{\tau_2}) \\ &= \frac{\langle \partial\gamma, b \rangle^{l_1+l_2-1}}{(l_1+l_2-1)!} \langle \partial\gamma, c_{0, \gamma}^{\tau_2} \rangle c_{0, \gamma}^{\tau_1} = 0. \end{aligned}$$

Let last equality follows from Corollary 6.12. Assume that $e_1^* = c^{-1}\partial\gamma$ is primitive in $H_1(L_p, \mathbb{Z})$ for some constant c . Extend it to a symplectic integral basis e_1^*, e_2^* and e_1, e_2 be the corresponding dual basis in $H^1(L, \mathbb{Z})$. For $k > 2$, the equation (21) follows from the induction hypothesis and the divisor axiom. For $k = 1$, the only trees contribute to $\mathfrak{c}'(1, d\gamma)(b, \dots, b)$ have one interior vertex v and $l+1$ exterior vertex (including the root), $l \geq 0$ or

only one vertex. Thus

$$\begin{aligned}
\mathfrak{c}'(1, d\gamma)(b) &= \sum_{l \geq 0} \int_0^1 c_{l, d\gamma}^\tau(b, \dots, b) d\tau \\
&= \sum_{l \geq 0} \frac{\langle d\partial\gamma, b \rangle^l}{l!} \int_0^1 c_{0, d\gamma}^\tau d\tau \\
&= (z^{\partial\gamma})^d \int_0^1 c_{0, d\gamma}^\tau d\tau
\end{aligned}$$

Together with Corollary 6.12, the transformation given by $F_{(\phi, p)}^{can}$ becomes

$$z_i \rightarrow z_i f, \text{ where } f = 1 + \begin{cases} 0 & \text{if } i = 1 \\ \text{power series in } z_1^c & \text{if } i = 2, \end{cases}$$

In terms of the above basis, ϖ is preserved under the transformation. \square

Fix $\lambda > 0$. There are finitely many of affine hyperplanes l_γ passing through U such that there exists γ such that $\mathcal{M}_\gamma(X, L_u) \neq \emptyset$ and $|Z_\gamma(p)| < \lambda$. A direct consequence of Theorem 6.13 is the following theorem.

Theorem 6.14. *Let u_+, u_- in a small enough neighborhood $U \ni p$ and a path ϕ connecting u_+, u_- falls in U . Then the transformation $F_{(\phi, p)}^{can}$ preserves the 2-form ϖ . In other words, we have $(F_{(\phi, p)}^{can})^* \varpi = \varpi$.*

Remark 6.15. 1. *We expect that the Theorem 6.13 is true for special Lagrangian surface of higher genus as well though our argument does not apply.*

2. *Similar result also appears in the work of Seidel (Section 11 [24]).*

6.4 Locally Around Type I_1 Singular Fibre

From this section we will focus on the case when X is a K3 surface with special Lagrangian fibration and L is a special Lagrangian torus fibre. Assume moreover that each of the singular fibre has a single simple node and we want to compute

$$F_\phi^{can} : H^1(L_{u_1}, \Lambda_+) \rightarrow H^1(L_{u_2}, \Lambda_+)$$

for a path ϕ connecting $u_1, u_2 \in B_0$ near a singularity $0 \in B$ of the affine structure.

Theorem 6.16. *Given $\lambda > 0$, there exists a neighborhood $\mathcal{U}_\lambda \ni 0$ such that for any path ϕ connecting u_1, u_2 totally contained in \mathcal{U}_λ , then*

1. *if ϕ does not intersect l_\pm , then $F_\phi^{can} \cong Id \pmod{T^\lambda}$.*
2. *if ϕ is homotopy to ϕ_\pm within \mathcal{U}_λ , then*

$$\begin{cases} F_{(\phi, p_\pm)}^{can}(z_1) = z_1 \\ F_{(\phi, p_\pm)}^{can}(z_2) = z_2(1 + T^{\pm\omega(\gamma_e)} z_1^{\pm 1})^{\pm 1} \end{cases} \pmod{T^\lambda},$$

Proof. Let U be a neighborhood of 0 and $X_U := p^{-1}(U)$ be the pre-image of the fibration, which has the same topological space as Ooguri-Vafa space. From the gradient estimate of holomorphic discs, a torus fibre L_u does not bound any holomorphic disc with symplectic area larger or equal to λ if we shrink U small enough. Then by Proposition 5.9, we may consider X_U instead of the whole K3 surface X .

It worth notice that for any torus fibre L , we have $H_2(X_U, L) \cong \mathbb{Z}$, which is generated by the Lefschetz thimble. Thus, the only holomorphic discs with small symplectic area have relative classes to be multiple cover of Lefschetz thimbles. In particular, this proves the first part of the theorem.

By Lemma 4.42 [15], there exists a 1-parameter family of hyperKähler structures (which we will denote the path by ϕ) connecting X_U and X_{OV} (after hyperKähler rotation), with the holomorphic volume form fixed. Moreover, the same underlying topological torus fibration is a special Lagrangian fibration with respect to any hyperKähler structures in the 1-parameter family. Assume that the size of the special Lagrangian fibre is small enough, so the hyperKähler structures of X_U and X_{OV} are closed enough to apply the Fukaya's trick (See Corollary 4.32 [15]). Without loss of generality, we will assume the endpoints u, u' of ϕ are closed enough so that Fukaya's trick may apply by shrinking U further if necessary.

By Corollary 5.6, we may choose $\tilde{u}, \tilde{u}' \in U$ near u, u' such that $(X, L_{\tilde{u}})$ and $(X, L_{\tilde{u}'})$ do not bound any holomorphic discs when the complex structure changing via the path ϕ_1 . It worth mentioning that such locus only depends on the holomorphic volume form of the elliptic fibration. Therefore, we have $F_{\phi_1}^{can} = Id$. Choose a path ϕ_2 (and ϕ_2') in U connecting u and \tilde{u} (u' and \tilde{u}' respectively) which do not intersect l_\pm (See Figure 3 below).

By the construction of U , there is only holomorphic discs in the relative class of Lefschetz thimble, ending on fibres in X_U and with symplectic area

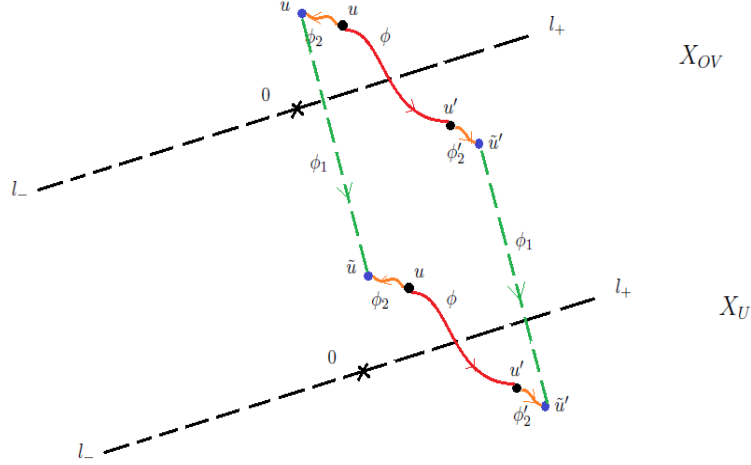


Figure 3: Connecting X_U to X_{OV} via hyperKähler manifolds.

less than λ . Therefore, we have

$$\begin{aligned} F_{\phi_2}^{can} &\equiv \text{Id} \pmod{T^\lambda} \\ F_{\phi'_2}^{can} &\equiv \text{Id} \pmod{T^\lambda}. \end{aligned}$$

from the first part of the theorem. Therefore,

$$\begin{aligned} F_\phi^{can} &= (F_{\phi'_2}^{can})^{-1} \circ (F_{\phi_1}^{can})^{-1} \circ F_{\phi_{OV}}^{can} \circ F_{\phi_1}^{can} \circ F_{\phi_2}^{can} \\ &\equiv F_{\phi_{OV}}^{can} \pmod{T^\lambda}. \end{aligned} \tag{22}$$

Equation (22) together with Theorem 6.9 proves the second part of the theorem. The last part of the theorem follows similarly. \square

Remark 6.17. *A priori, we can't exclude the situation that a tropical/holomorphic disk in the relative class of $k\gamma_e$ with stop in $u \in B_0$ near 0 when k is large. Thus, it might be too optimistic to expect a universal \mathcal{U} such that the following holds for all $\lambda > 0$.*

6.5 Open Gromov-Witten Invariants Revisited

The wall-crossing phenomenon of Maurer-Cartan elements motivate the definition of another open Gromov-Witten invariant $\hat{\Omega}'(\gamma; u)$, which satisfies the

Kontsevich-Soibelman wall-crossing formula and share most of the properties as $\tilde{\Omega}(\gamma; u)$ defined in [15].

Lemma 6.18. *Assume that $\Omega : H_2(X, \mathbb{Z})/\mathbb{Z}[L_u] \rightarrow \mathbb{C}$ is injective. Let l be an affine line segment in B_0 such that L_u bounds holomorphic discs in the relative class γ for $u \in l$. Fix a $\lambda > 0$. Assume that $u \in l$ does not fall on $W'_{d\gamma}$ for any $d \in \mathbb{Z}$ such that $d|Z_\gamma(u)| < \lambda$. Then there exists a small enough neighborhood U_λ of p such that for $u'_\pm, u''_\pm \in U_\lambda$ on different side of l and paths $\phi_\pm \in U_\lambda$ such that $\phi_\pm(0) = u'_\pm$ and $\phi_\pm(1) = u''_\pm$, we have*

$$F_{(\phi_+, u)}^{can} = F_{(\phi_-, u)}^{can} \pmod{T^\lambda}.$$

Proof. From the proof of Theorem 6.13, the transformation $F_{\phi_\pm}^{can}$ acts as

$$F_{\phi_\pm}^{can} : z^{\partial\gamma'} \mapsto z^{\partial\gamma'} f_{\gamma, \pm}^{\langle \partial\gamma, \partial\gamma' \rangle},$$

for some function $f_{\gamma, \pm} \in \Lambda^S / F^\lambda \Lambda^S[[z^{\partial\gamma}]]$. Let ϕ' (and ϕ'') be a path connecting u'_+, u'_- (and u''_+, u''_- respectively) in U_λ as shown in the picture below. By shrinking U_λ if necessary, we may assume that there are only finitely many affine lines $\lambda_{\gamma'}$ in U_λ such that the torus fibres above those affine lines can bound holomorphic discs in relative class γ' with $|Z_{\gamma'}(u')| < \lambda$ for some $u' \in U_\lambda \cap l_{\gamma'}$. Thus $F_{\phi'}, F_{\phi''}$ can be expressed as product of elementary transformations. The assumption that u doesn't fall in $W'_{d\gamma}$ for any $d \in \mathbb{Z}$ guarantees that there are no terms of the form $z^{d\gamma}$, $d \in \mathbb{Z}$ in those slab functions associated to the above elementary transformations. Let $d \in \mathbb{N}$ be the smallest integer such that

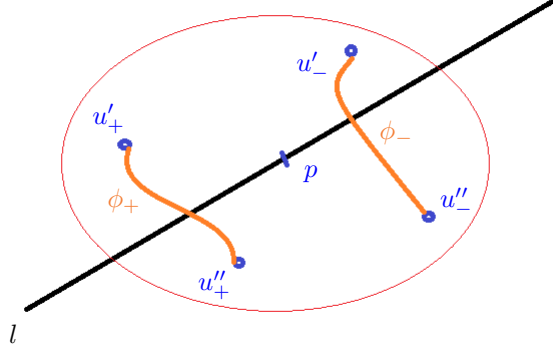
$$f_{\gamma, +} \neq f_{\gamma, -} \pmod{z^{d\gamma}}.$$

Now from Theorem 6.13, each term in

$$(F_{\phi_-}^{can})^{-1} \circ F_{\phi}^{can} \circ F_{\phi''}^{can} \circ F_{\phi_+}^{can} \tag{23}$$

can be expressed in terms of the product of elementary transformation (see Remark 4.22). Then from Baker-Campbell-Hausdorff formula, the coefficient of $z^{d\partial\gamma}$ in (23) is non-zero. This contradicts to Theorem 5.12. \square

Given an affine line l_γ labeled by γ and an unbounded increasing sequence $\lambda_i \in \mathbb{R}_{>0}$, there exists $u_{i,+}, u_{i,-} \in U_{\lambda_i}$ on different side of l and a path ϕ_i



in U_λ connecting $u_{i,+}, u_{i,-}$ such that $\int_\gamma(\omega - i\text{Im}\Omega)$ is decreasing, then the transformation F_ϕ^{can} acts as

$$F_\phi^{\text{can}} : z^{\partial\gamma'} \mapsto z^{\partial\gamma'} f_{i,\gamma}^{\langle\gamma', \gamma\rangle},$$

with $f_{i,\gamma} \in \Lambda^{\mathcal{S}_\gamma}[[z^{\partial\gamma}]]/F^{\lambda_i}\Lambda^{\mathcal{S}_\gamma}$.

Definition 6.19. Assume that $\Omega : H_2(X, \mathbb{Z})/\mathbb{Z}[L_u] \rightarrow \mathbb{C}$ is injective.

1. With the notation above, the inverse limit $\lim_{\leftarrow} f_{i,\gamma}$ exists and is independent of various choices made in the construction. In other words, we can associate a power series

$$f_\gamma(u) := \lim_{\leftarrow} f_{i,\gamma} \in \Lambda^{\mathcal{S}_\gamma}/F^{\lambda_i}\Lambda^{\mathcal{S}}[[z^{\partial\gamma}]]$$

for each $u \in l$.

2. Given $u \in B_0$ and $\gamma \in H_2(X, L_u)$. We will compute $f_\gamma(u)$ in X_ϑ , $\vartheta = \text{Arg}Z_\gamma(u)$, and define the open Gromov-Witten invariants $\tilde{\Omega}'(\gamma; u)$ by the formula

$$f_\gamma(u) = \sum_{d \geq 1} d\tilde{\Omega}'(d\gamma; u)(z^{\partial\gamma} T^{Z_\gamma(u)})^d. \quad (24)$$

Remark 6.20. If we loose the generic condition, we might only be able to define $\tilde{\Omega}(\gamma; u)$ for $\gamma \in H_2(X, L_u)/\sim$, where $\gamma \sim \gamma'$ if $\gamma \in \gamma' + \text{Ker}\Omega$.

A direct consequence of Lemma 6.18 is the following:

Lemma 6.21. *Assume that the affine line segment from u_1 to u_2 does not intersect $W'_{d\gamma}$ for any $d \in \mathbb{Z}$ and $d|Z_\gamma| < \lambda$. Then we have*

$$T_{u_1, u_2} f_\gamma(u_1) = f_\gamma(u_2) \pmod{z^{d\partial\gamma}}. \quad (25)$$

In particular, this implies the following basic properties of the open Gromov-Witten invariants $\tilde{\Omega}'(\gamma; u)$.

Proposition 6.22. *Assume that X is a generic elliptic K3 surface and $u \notin W'_\gamma$, then $\tilde{\Omega}'(\gamma; u)$ is well-defined. Moreover, we have the following properties:*

1. *If there exists a path connecting $u, u' \in B_0$ does not pass through W'_γ , then*

$$\tilde{\Omega}'(\gamma; u) = \tilde{\Omega}'(\gamma; u').$$

2. *Let $u \in W'_\gamma$ generic and let u_+, u_- be on the different side of W'_γ near u . Assume that there exists no $\gamma_i \in H_2(X, L_u)$ such that $\sum_i \gamma_i = \gamma$, $\text{Arg}Z_{\gamma_i} = \text{Arg}Z_\gamma$ and $\tilde{\Omega}'(\gamma; u) \neq 0$, then*

$$\tilde{\Omega}'(\gamma; u_+) = \tilde{\Omega}'(\gamma; u).$$

3. *(Reality condition) $\tilde{\Omega}'(\gamma; u) = \tilde{\Omega}'(-\gamma; u)$.*

Proof. If $\text{Arg}Z_\gamma(u) = \text{Arg}Z_\gamma(u')$, then the first property follows from Lemma 6.21 by dividing the path connecting $u, u' \in B_0$ into segments smaller enough. In general, the first property follows from the similar discussion in section 6.6. From the assumption that $u_0 \in W'_\gamma$ generic, we may assume that u_0 doesn't fall on $W''_{\gamma'}$ with $|Z_{\gamma'}| < |Z_\gamma|$. Thus, for u, u' closed enough to u_0 , there exists a path ϕ connecting the two points without touch the walls described. The second property follows from the similar proof of Lemma 6.18. The third property follows from checking the orientation of the moduli spaces of $\mathcal{M}_{k, \gamma}(\mathfrak{X}, L_u)$ and $\mathcal{M}_{k, -\gamma}(\mathfrak{X}, L_u)$, which is similar to the proof of the reality condition in [15]. □

Now we are ready for the weak correspondence theorem

Theorem 6.23. *Assume that X is a generic elliptic K3 surface. Let $\gamma \in H_2(X, L_u)$ and $\tilde{\Omega}'(\gamma; u) \neq 0$. Then there exists a tropical disc (ϕ, T, w) with stop at u and $[\phi] = \gamma$.*

Proof. Given $\gamma \in H_2(X, L_u)$ and $\tilde{\Omega}'(\gamma; u) \neq 0$, this implies that $\tilde{\Omega}'(\gamma; u') \neq 0$ for u' in a neighborhood of u . Let u' moves along l_γ in the direction such that $|Z_\gamma(u')|$ is decreasing. From Lemma [15], $|Z_\gamma|$ will decrease to zero on l_γ at a point u_0 . If $\tilde{\Omega}'(\gamma; u')$ is constant on l_γ between u and u' , then L'_u is a singular fibre. In particular, if X has only I_1 -type singular fibre then γ is a multiple of Lefschetz thimble. If $\tilde{\Omega}'(\gamma; u')$ jump at some point u_1 on l_γ , then $u' \in W'_\gamma$. From the second part of Proposition 6.22, there exists γ_i such that $\gamma = \sum_i \gamma_i$ and $\tilde{\Omega}'(\gamma_i; u_1) \neq 0$. Let

$$l_{\gamma_i}^- := \{u' \in l_{\gamma_i} | u' \text{ near } u_1 \text{ and } |Z_{\gamma_i}(u')| < |Z_{\gamma_i}(u_1)|\}.$$

Then union of $l_{\gamma_i}^-$ and the segment of l_γ between u and u_1 gives an image of tropical rational curve around u_1 . Replace γ_i by γ and repeat the argument. Notice that $|Z_{\gamma_i}| < |Z_\gamma|$ and the procedure stops after finitely many steps by Gromov compactness theorem. The union of all local tropical rational curves gives an image of a tropical disc ends at u . Notice that the decomposition $\gamma = \sum_i \gamma_i$ at each vertex might not be unique and might lead to different (image of) tropical discs. \square

Now we want to prove that the open Gromov-Witten invariants $\{\tilde{\Omega}'(\gamma; u)\}$ satisfies the Kontsevich-Soibelman wall-crossing formula.

Theorem 6.24. *Assume that X is a generic elliptic K3 surface, then the open Gromov-Witten invariants $\tilde{\Omega}'(\gamma; u)$ satisfy the analogue of Theorem 4.14*

Proof. Theorem 5.12 is the Floer theoretic analogue of Theorem 4.15. The rest of the proof is similar to the proof of Theorem 4.14. \square

From the discussion in previous sections, the two open Gromov-Witten invariants $\tilde{\Omega}(\gamma; u)$ and $\tilde{\Omega}'(\gamma; u)$ share many properties. Actually, they are indeed the same invariants via the analogue of divisor axiom for Gromov-Witten invariants.

Theorem 6.25. *Assume that X is a generic elliptic K3 surface. Let $\gamma \in H_2(X, L_u)$ and assume that $u \notin W'_\gamma$, then $\tilde{\Omega}(\gamma; u)$ and $\tilde{\Omega}'(\gamma; u)$ are both well-defined and*

$$\tilde{\Omega}(\gamma; u) = \tilde{\Omega}'(\gamma; u).$$

We will postpone the proof to Section 6.6. Similar to the tropical story, we have the analogue definition of wall of marginal stability.

Definition 6.26. *The wall of marginal stability for holomorphic discs are defined by*

$$W_\gamma := \{u \in W'_\gamma \mid \Delta\tilde{\Omega}(\gamma) \neq 0\}.$$

For $u \in W'_\gamma \setminus W_\gamma$, we will define $\tilde{\Omega}(\gamma; u)$ by the natural continuous extension.

Remark 6.27. *Similar proofs shows that Theorem 6.25 holds when $u \notin W_\gamma$.*

This will lead us to the main theorem of the paper, which matches the tropical discs counting with the open Gromov-Witten invariants.

Theorem 6.28. *Under the same assumption¹³, then*

$$\tilde{\Omega}(\gamma; u) = \tilde{\Omega}^{trop}(\gamma; u). \quad (26)$$

Proof. First we consider the case, when X is generic. Given a pair (u, γ) , $u \in B_0$ and $\gamma \in H_2(X, L_u)$ with $u \notin W_\gamma \cup W_\gamma^{trop}$, we associate an integer $n(\gamma; u)$ to be the maximal number of internal vertices of the image of tropical discs stop at u such that its relative class is γ . Explicitly,

$$n(\gamma; u) := \max\{|(C_0^{ext}(\bar{T})| - 1|(\phi, T, w) \text{ a tropical disc stop at } u \text{ with } [\phi] = \gamma.\}$$

From Lemma 4.10, the number $n(\gamma; u)$ is finite. We will first prove the theorem by induction on $n(\gamma; u)$.

First we assume that $n(\gamma; u) = 0$. From the proof of Theorem 6.23, we have that γ is the parallel of the multiple of Lefschetz thimble from a singularity and

$$\tilde{\Omega}(\gamma; u) = \tilde{\Omega}(\gamma; u'),$$

for any u closed enough to the singularity. Then the theorem follows from Definition 4.8, Theorem 6.3 and Theorem 6.25.

Now assume that the theorem is true for all pairs (u', γ') with $n(\gamma'; u') \leq n$ and now we want to prove the statement when $n(u; \gamma) = n+1$. There exists an affine line l_γ passing through u such that Z_γ has constant phase along l_γ . Move u along l_γ in the direction such that $|Z_\gamma|$ is decreasing. From the assumption l_γ will intersect W_γ^{trop} and the theorem follows from Theorem 6 in [11], Theorem 5.12 and the induction hypothesis. Notice that in the mean time, we also proved that the wall of marginal stability of holomorphic discs and tropical discs are the same, i.e, $W_\gamma = W_\gamma^{trop}$. For general case, there exists u' near u such that

¹³Notice that we don't need the generic assumption on elliptic K3 surface X .

1. $\tilde{\Omega}^{trop}(\gamma; u') = \tilde{\Omega}^{trop}(\gamma; u)$.
2. all tropical discs contribute to $\tilde{\Omega}^{trop}(\gamma; u')$ have no internal vertex maps to singularities.

These condition are preserved as we slightly deform X to a generic elliptic K3 surface X' . Let $\tilde{\Omega}_{X'}^{trop}(\gamma; u')$ (and $\tilde{\Omega}_{X'}(\gamma; u')$) denotes the corresponding tropical discs counting invariants (and open Gromov-Witten invariants) on X' . Then

$$\tilde{\Omega}(\gamma; u) = \tilde{\Omega}(\gamma; u') = \tilde{\Omega}_{X'}(\gamma; u') = \tilde{\Omega}_{X'}^{trop}(\gamma; u') = \tilde{\Omega}^{trop}(\gamma; u') = \tilde{\Omega}^{trop}(\gamma; u).$$

□

6.6 Open Gromov-Witten Invariants via Real Noether-Lefschetz Theory

The Lemma 6.18 and Definition 6.19 can actually be generalized to the following setting following the same argument. Consider \mathcal{M} be the moduli space of the marked K3 surfaces together with special Lagrangians. Let \mathbb{L}_{K3} be the K3 lattice. Explicitly, \mathcal{M} is the set of 5-tuples,

$$\mathcal{M} = \{(X, \omega, \Omega, \alpha, L)\} / \sim, \quad (27)$$

where X is a K3 surface with a hyperKähler pair (ω, Ω) , $\alpha : H^2(X, \mathbb{Z}) \cong \mathbb{L}_{K3}$ is a marking and L is a smooth (oriented) special Lagrangian torus with respect to the pair (ω, Ω) , i.e. $\omega|_L = \text{Im}\Omega|_L = 0$. Two 5-tuples $(X, \omega, \Omega, \alpha, L)$ and $(X', \omega', \Omega', \alpha', L')$ are equivalent if and only if there exists a diffeomorphism $f : X \rightarrow X'$ such that $f^*\omega' = \omega$, $f^*\Omega' = \Omega$, $\alpha' \circ f^* = \alpha$ and $f(L) = L'$. The moduli space \mathcal{M} is an infinite disjoint union of smooth oriented manifolds of real dimension 59[16]. It worth mentioning that the boundary of one irreducible component of \mathcal{M} may fall in another irreducible component.

We will abuse notation and denote \mathcal{M} for one of its components and write (X, L) for the 5-tuple by view the rest as part of the data in X for simplicity. Now assume that there is a pair $u = (X_0, \omega_0, L_0) \in \mathcal{M}$ such that L_0 bounds a holomorphic disc in the relative class $\gamma \in H_2(X_0, L_0)$. Then there is a submersion from a small neighborhood \mathfrak{U} of \mathcal{M} containing (X_0, L_0) such that $\int_{\gamma(X, L)} \omega_X \neq 0$, for $(X, L) \in \mathfrak{U}$. Thus there is a well-defined map from \mathfrak{U} to S_ϑ^1 given by,

$$\text{Arg}_\gamma = \text{Arg}(Z_\gamma) : \mathfrak{U} \rightarrow S_\vartheta^1,$$

where

$$Z_\gamma(X, L) = -i \int_{\gamma(X, L)} \text{Im} \Omega_X + i \omega_X,$$

for $(X, \omega, \Omega, \alpha, L) \in \mathfrak{U}$. The map Arg_γ is a locally a submersion and thus the fibre of Arg_γ over $1 \in S^1$, denoted by NL_γ , is a real codimension one submanifold of \mathcal{M} . NL_γ can be viewed as the real analogue of the Noether-Lefschetz divisor. We will call them Noether-Lefschetz walls because they are codimension one submanifolds, which separates a tubular neighborhood of locus of \mathcal{M} where γ can be represented as holomorphic discs into two parts. Since the phase S_θ^1 is viewed as the unit circle in the complex plane and thus is naturally oriented, the Noether-Lefschetz walls NL_γ are also oriented.

Now assume there are a sequence of pairs of points $\{u_i^\pm = (X_i^\pm, \omega_i^\pm, \Omega_i^\pm, \alpha_i^\pm, L_i^\pm)\}$ in \mathcal{M} converging to u but fall on different sides of NL_γ . Choose paths ϕ_i connected u^\pm in a neighborhood in u . Following the same argument in Lemma 6.18 and Definition 6.19, we get a power series $f_\gamma(u) \in \Lambda[[z^{\partial\gamma}]]$, for $u \in NL_\gamma$. Same argument in Lemma 6.18 shows that $f_\gamma(u)$ is independent of the choice of the sequences u_i^\pm . In particular, we are interested in the following two cases (the paths both go in the direction such that $\text{Arg} Z_\gamma$ is decreasing):

1. $X_i^+ = X_i^-, \omega_i^+ = \omega_i^-, \Omega_i^+ = \Omega_i^-, \alpha_i^+ = \alpha_i^-, L_i^+$ and L_i^- are two special Lagrangians on different sides of the affine line l_γ .
2. $X_i^+ = X_i^-, \alpha_i^+ = \alpha_i^-, L_i^+ = L_i^-$, and

$$\begin{aligned} \omega_i^\pm &= \text{Im}(e^{\mp i\epsilon_i}(\text{Im} \Omega_0 + i\omega_0)), \\ \Omega_i^\pm &= \text{Re} \Omega_0 + i\text{Re}(e^{\mp i\epsilon_i}(\text{Im} \Omega_0 + i\omega_0)), \end{aligned}$$

for some $\epsilon_i \searrow 0$. In other words, the path ϕ_i is an arc contained the S^1 -family of hyperKähler structures described in Section 6.1 with two end points on the different side of NL_γ .

From the discussion above, the power series f_γ constructed from the two families above are the same.

Proof. (of Theorem 6.25) Let u_\pm be two points on the different side of l_γ and ϕ is a path connecting u_\pm . We may take u_\pm be close enough to l_γ such there is only relative class $d\gamma$, where $d \in \mathbb{N}$ and γ primitive, can be realized

as holomorphic discs with boundary on $L_{\phi(t)}$ for some t and with symplectic area less than λ for a given $\lambda > 0$. By Proposition 2.12, we have

$$F_\phi(b) = \sum_{k,d} \mathfrak{c}'(k, d\gamma)(b) T^{d\omega(\gamma)},$$

where $\mathfrak{c}(k, \gamma)$ is defined in Theorem 2.9. Let e_1, e_2 be an integral basis of $H_1(L_u)$ such that $\langle \gamma_1, \gamma_2 \rangle = 1$ and $\partial\gamma = ce_1, c \in \mathbb{N}$. To show the theorem, it suffices to show that

$$-d\tilde{\Omega}(d\gamma; u) z_1^{cd} e_2 = \sum_{k \geq 1} \mathfrak{c}'(k, d\gamma)(b),$$

From the proof of Theorem 6.13, we have $\mathfrak{c}(k, d\gamma)(b, \dots, b) = 0$, for $k \geq 2$. For $k = 1$, we have

$$\mathfrak{c}'(1, d\gamma)(b) = (z^{\partial\gamma})^d \int_0^1 c_{0,d\gamma}^\tau d\tau$$

Since $\mathfrak{c}'(1, d\gamma)(b) = F_\phi(b)$ does not depends on the choice of the path ϕ in \mathcal{M} , we may change it to the arc in the S^1 -family in the twistor line described above. By Lemma 6.11, we have

$$\begin{aligned} \mathfrak{c}'(1, d\gamma)(b) &= (z^{\partial\gamma})^d \int_0^1 c_{0,d\gamma}^\tau d\tau \\ &= -(z^{\partial\gamma})^d \tilde{\Omega}(d\gamma; u) de_2. \end{aligned}$$

□

A Existence of Quadratic Refinement

Proposition A.1. *Let Γ' be the local system generated by the relative classes which can be realized as tropical discs with respect to some $\vartheta \in S^1$. There exists a unique well-defined quadratic refinement defined on Γ' such that the value is -1 for every parallel transport of the Lefschetz thimble.*

Proof. We will actually prove that there exists an integer-value function $C : \Gamma_g := \cup_{u \in B_0} H_1(L_u) \rightarrow \mathbb{Z}$ satisfying

1. For any $\partial\gamma_1, \partial\gamma_2 \in \Gamma_g$, we have

$$C(\partial\gamma_1 + \partial\gamma_2) = C(\partial\gamma_1)C(\partial\gamma_2) + \langle \partial\gamma_1, \partial\gamma_2 \rangle \pmod{2}.$$

2. If $\partial\gamma \in \Gamma_g$ is a parallel transport of vanishing cycle from an I_1 -type singular fibre then $C(\partial\gamma) = 1$.

Then any $\gamma \in \Gamma$, we can define

$$c(\gamma) = (-1)^{C(\partial\gamma)}.$$

Assume there are two tropical discs images represents the same relative homology class. Without lose of generality, we may assume that there exists integers $k_i, k \in \mathbb{Z}$ and parallel transport of Lefschetz thimbles $\gamma, \gamma_i \in H_2(X, L_u)$ such that

$$\sum_i k_i \gamma_i = k\gamma.$$

It suffices to prove that the quadratic refinement satisfies

$$C(\sum_i k_i \partial\gamma_i) = C(k\partial\gamma) \pmod{2}, \quad (28)$$

where $k_i, k \in \mathbb{Z}$ and $\partial\gamma_i, \partial\gamma \in H_1(L_u)$ are parallel of vanishing cycles. Notice that $c(\partial\gamma_i) = 1$ and $c(\partial\gamma) = 1$. By choose a symplectic basis for $H_1(L_u)$ and set

$$\partial\gamma_i = (v_i^1, v_i^2), \quad \partial\gamma = (v^1, v^2).$$

Then we have

$$\begin{aligned} c(k_i \partial\gamma_i) &= 1 - (1 - k_i v_i^1)(1 - k_i v_i^2) \\ c(k_i \partial\gamma) &= 1 - (1 - \sum_i k_i v_i^1)(1 - \sum_i k_i v_i^2). \end{aligned}$$

The equation (28) is reduced to the following identity in \mathbb{Z}_2

$$1 - \left(1 - \sum_i k_i v_i^1\right) \left(1 - \sum_i k_i v_i^2\right) = \sum_k \left[1 - (1 - v_k^1)(1 - v_k^2)\right] + \sum_{i \neq j} v_i^1 v_j^2.$$

□

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